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BRIEF

COLLEGE ALGEBRA

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REVISED EDITION

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PREFACE

THIS TEXT was designed to permit students of proper preparation to obtain speedy contact with the most useful and interesting parts of college algebra. The book offers a rapid* but coherent review of the elements of algebra followed by a normal leisurely treatment of the truly collegiate parts of *college algebra.* The text was planned as a basis for an efficient first course in college algebra for a class where the typical student has had an orthodox third semester of high school algebra. The relative brevity of the book is due solely to the concise treatment of the most elementary topics and the moderately abbreviated discussion of intermediate content. In the remainder of the text, the usual topics of college algebra are presented in complete detail without any special acceleration. In addition to the customary classical material, the book includes a substantial body of content† particularly appropriate for students of engineering, business administration, the various social sciences, and other fields having contact with statistics. This content involves a discussion of logarithmic and semilogarithmic graphing, the method of least squares for solving linear equations, with applications to curve fitting, and related statistical topics through the stage of simple correlation.

In the preparation of this *Revised Edition,* all major features of the author's *Brief College Algebra* were preserved.‡ However, in comparison with that book, this *Revised Edition* presents many variations in details.

* If a text is desired which offers a more leisurely treatment of intermediate algebra as a preliminary to the presentation of classical college algebra, see *College Algebra, Third Edition,* by WILLIAM L. HART, D. C. HEATH AND COMPANY.

† This material was presented as a special feature of the first edition of this text.

‡ A minor exception is that the present text does not include the chapter on the Mathematics of Investment which occurred in the original edition. For a text including such a chapter, see HART's *College Algebra, Third Edition.*

The whole book has been reset in an appealing new format on a large page.

As a consequence of classroom experience with the first edition, a few changes have been made in the location of optional chapters and certain supplementary topics, to aid the teacher in maintaining continuity when it is desired to present a minimum course.

Fresh exercises are included, except in certain places where the problems were of simple drill type or of a unique character which dictated their retention.

SPECIAL FEATURES

The introductory material. The first few chapters exhibit the complete skeleton of elementary algebra with emphasis on definitions, terminology, and the logical sequence of topics, and with a collegiate view of the objectives. Sufficient problems are given in the associated exercises to offer material for a leisurely review if the class requires it. Various sets of miscellaneous problems, if taken alone, provide a basis for a very rapid review.

The advanced chapters. A high standard of logical accuracy is maintained throughout. However, the discussions and proofs are phrased so that they should be readily understood by the typical student for whom the book is intended.

The character of the exercises. The problem material is exceptionally abundant and in each exercise the examples are arranged approximately in order of increasing difficulty.

Supplementary material. Content which is not essential in a minimum course is segregated into obviously independent chapters or is clearly labeled with a black star, ★, or the word *supplementary.* The omission of such material in whole or in part does not disturb the continuity of the text. Moreover, an effort was made to locate optional material in such a way that if it were omitted, this action would not give a superficially discontinuous appearance to the remaining content.

Orientation for later mathematics. At appropriate places, the text gives emphatic attention to the special needs of technical students and others who will continue their study of mathematics at least through calculus.

Terminology. Fundamental definitions and the useful technical language associated with functions, equations, and analytic geometry are introduced and emphasized wherever possible, so that the student will gain early familiarity with these important parts of the vocabulary of collegiate mathematics.

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CONTENTS

CHAPTER ONE

Elementary Topics

1. Real numbers. The numbers of arithmetic and the elementary parts of algebra are called *real* numbers. They are classed as either *positive, negative,* or *zero.* Later, we shall introduce *so-called* **imaginary** numbers as the square roots of negative numbers. Until then, any number referred to will be a real number. In the case of a positive number, it is customary to omit writing its sign, +, except for emphasis. For contrast with *explicit* numbers like 3, − 5, etc., we agree that number symbols such as a, b, x, and y will be called *literal* numbers. In this book, any letter which is introduced without a qualifying description will represent a *number.*

ILLUSTRATION 1. 3, 0, and $-\frac{2}{5}$ are real numbers. We read "− 5" as *minus five.* The positive number 6 may be written + 6 for emphasis.

2. The fundamental operations of algebra are addition, subtraction, multiplication, and division.

3. Addition. The result of adding two or more numbers is called their **sum.** We use the plus sign, +, to indicate addition.

Addition is commutative, or the sum of two numbers is the same in whatever order they are added.

ILLUSTRATION 1. $\quad 3 + 5 = 5 + 3 = 8. \qquad a + b = b + a.$

Addition is associative, or, the sum of three or more numbers is the same in whatever order they are grouped in adding.

ILLUSTRATION 2.
$$a + b + c = a + (b + c)$$
$$= b + (a + c) = (a + b) + c.$$

4. Subtraction. To subtract b from a means to find the number x which, when added to b, will yield a. We employ the minus sign, −, to indicate subtraction. By the **difference** of two numbers a and b we mean the result of subtracting the *second* number from the *first.*

ILLUSTRATION 1. The difference of 7 and 3 is (7 − 3) or 4. If x is the difference of a and b, then $x = a - b$ and $a = b + x$.

5. Multiplication. The result of multiplying two or more numbers is called their **product,** and each of the given numbers is called a **factor** of their product. To indicate multiplication we use a cross, $\times$, or a dot between the numbers, or merely write them side by side with no signs of operations between them.

ILLUSTRATION 1. $15ab$ means $15 \cdot a \cdot b$ and is read "*fifteen a, b.*"

Multiplication is commutative, or the product of two numbers is the same in whatever order they are multiplied.

ILLUSTRATION 2. $7 \times 3 = 3 \times 7 = 21. \qquad ab = ba.$

Multiplication is associative, or the product of three or more numbers is the same in whatever way they are grouped.

ILLUSTRATION 3. $5 \times 7 \times 6 = 7 \times (5 \times 6) = (5 \times 7) \times 6.$

$$abc = (ab)c = (ac)b = a(bc).$$

Multiplication is distributive with respect to addition:

$$a(b + c) = ab + ac.$$

6. Division. To divide a by b, where b is *not zero,* means to find the number x such that $a = bx$. We call a the **dividend,** b the **divisor,** and x the **quotient.** We denote the quotient by $a \div b$, or $\frac{a}{b}$, or a/b. The fraction a/b is read "*a divided by b,*" or "*a over b.*" In a/b, we call a the **numerator** and b the **denominator;** also, a and b are sometimes called the **terms** of the fraction. The fraction a/b, or the quotient $a \div b$, is frequently referred to as the **ratio** of a to b.

7. Operations involving zero. We have mentioned that the operation $a \div b$ is not defined when $b = 0$; that is, **division by zero is not allowed.** However, no exception arises in multiplying by zero, adding zero, subtracting zero, or in dividing zero by some other number. Thus, if N is any number,

$$N + 0 = N; \quad N - 0 = N; \quad (N \times 0) = 0.$$

If N is not 0, then $\frac{0}{N} = 0$, because $0 = 0 \times N$.

Note 1. Contradictions arise when an attempt is made to define division by zero. Thus, if we were to define $5/0$ to be the number x which, when multiplied by zero, will yield 5, we would obtain $5 = 0 \cdot x$, or $5 = 0$, which is contradictory.

8. The absolute value of a positive number or of zero is the number itself. The absolute value of a *negative* number is the given number with its sign *changed*. The absolute value of a number is also called its *numerical value*. The absolute value of a is represented by $|a|$.

ILLUSTRATION 1. The absolute value of 5 is 5. The absolute value of -5 also is 5. $|-3| = 3$.

9. Laws of signs. The following rules govern the use of plus and minus signs in elementary operations. In finding the product of two numbers, or the quotient when one number is divided by another, give the result a *plus sign* if the numbers have *like signs;* give the result a *minus sign* if the numbers have *unlike signs.*

ILLUSTRATION 1. $(-6)\cdot(-7) = +42.$ $\dfrac{-40}{+4} = -10.$ $\dfrac{4}{-3} = -\dfrac{4}{3}\cdot$

To add two numbers having like signs, *add the absolute values* of the numbers and attach their *common sign.* To add two numbers having *unlike* signs, subtract the *smaller absolute value from the larger* and prefix the sign of the number with the larger absolute value.

ILLUSTRATION 2. $(-5) + (-7) = -12.$ $(-8) + 3 = -5.$

Placing a minus sign before a number is equivalent to multiplying the number by -1, and the result is called the **negative** of the given number. Hence, the negative of a positive or of a negative number is obtained by changing its sign.

ILLUSTRATION 3. The negative of 5 is -5. The negative of -3 is $-(-3)$ or $+3$. The negative of x is $-x$. The negative of 0 is 0 itself.

To subtract a number, *change its sign and add the resulting number.* That is, to subtract a number, add its *negative.*

ILLUSTRATION 4. $6-(-3) = 6+3 = 9.$ $-6-(+3) = -6-3 = -9.$

10. Terms. An expression like $(3x - 2y + 5)$ is referred to as a *sum*, or sometimes as an *algebraic sum.* In a sum, each part, with the sign which precedes it, is called a *term* of the expression.

ILLUSTRATION 1. In $(3x - 5y + 7)$, the terms are $3x$, $-5y$, and 7.

11. Parentheses, ( ), brackets, [ ], braces, { }, and the vinculum, $\overline{\quad\quad}$, are symbols of grouping used to indicate terms whose sum is to be treated as a single number expression. To remove, or to insert, parentheses preceded by a *plus* sign, rewrite the included terms

without altering their signs. To remove, or to insert, parentheses preceded by a *minus* sign, rewrite the terms involved with their signs *changed.* If a symbol of grouping incloses one or more other symbols of grouping, remove them by removing the innermost symbol first. and so on until the last one is removed.

ILLUSTRATION 1. $-3+5b-x=-(3-5b+x)$.

ILLUSTRATION 2. $-\{3y-[(x+2)-4z]\}=-\{3y-[x+2-4z]\}$
$$=-\{3y-x-2+4z\}=-3y+x+2-4z.$$

12. A fundamental principle for fractions. The value of a fraction is not changed if *both numerator and denominator are multiplied* (*or divided*) *by the same number, not zero.*

ILLUSTRATION 1. $\dfrac{3}{7}=\dfrac{3\times 4}{7\times 4}=\dfrac{12}{28}.$ $\qquad \dfrac{24}{36}=\dfrac{24\div 12}{36\div 12}=\dfrac{2}{3}.$

To reduce a fraction to **lowest terms,** we divide out *all common factors of the numerator and denominator.*

ILLUSTRATION 2. $\dfrac{15ad}{21cd}=\dfrac{3\cdot 5ad}{3\cdot 7cd}=\dfrac{5a}{7c}.$ (Divide out $3d$)

If the numerator (or denominator) of a fraction is multiplied by -1, the sign before the fraction must be *changed.*

ILLUSTRATION 3. $\dfrac{-5}{7}=-\dfrac{(-1)(-5)}{7}=-\dfrac{5}{7}.$ $\qquad \dfrac{4}{-a}=-\dfrac{4}{a}.$

13. Multiplication and division of fractions. The product of two fractions is a fraction whose numerator is the *product of the numerators* of the given fractions, and whose denominator is the *product of the given denominators.* To divide one fraction by another, *invert the divisor and multiply* the dividend by this inverted divisor.

ILLUSTRATION 1. $\left(\dfrac{4}{5}\div\dfrac{3}{7}\right)=\dfrac{4}{5}\cdot\dfrac{7}{3}=\dfrac{28}{15}.$ $\qquad (2\tfrac{3}{5}\div 4\tfrac{2}{5})=\dfrac{\frac{13}{5}}{\frac{22}{5}}=\dfrac{13}{22}.$

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}. \qquad \frac{\frac{a}{b}}{\frac{c}{d}}=\frac{a}{b}\cdot\frac{d}{c}=\frac{ad}{bc}.$$

It is frequently useful to recall that any number can be expressed as a fraction whose denominator is 1.

ILLUSTRATION 2. $7\left(\dfrac{5}{6}\right)=\dfrac{7}{1}\cdot\dfrac{5}{6}=\dfrac{35}{6}.$ $\qquad \left(6\div\dfrac{5}{7}\right)=\dfrac{6}{1}\cdot\dfrac{7}{5}=\dfrac{42}{5}.$

To multiply a fraction by a number, multiply the numerator by the number. To divide a fraction by a number, multiply the denominator by the number.

ILLUSTRATION 3. $c\left(\frac{a}{b}\right) = \frac{c}{1}\cdot\frac{a}{b} = \frac{ac}{b}.$ $\left(\frac{a}{b} \div c\right) = \left(\frac{a}{b} \div \frac{c}{1}\right) = \frac{a}{b}\cdot\frac{1}{c} = \frac{a}{bc}.$

EXERCISE 1

Find the sum, difference, and product of the two numbers, and the quotient of the first number divided by the second number.

1. 36 and 21.
2. 16 and 28.
3. -15 and 20.
4. $-.4$ and $-.12$.
5. 0 and 12.
6. 0 and -5.

Remove all signs of grouping and combine terms.

7. $2x - (4x - 1)$.
8. $-[4a - (2a + 3)]$.
9. $-(a - 2) - [2a - (a - 3)]$.
10. $-\{a - [a - (2a - 7)]\}$.
11. $-(2x - 3y) + [2 - (5x - y)] - (x + 7y - 8)$.
12. $3(2x - 5y) - 6(x - 2y)$.
13. $-5(3 - x) + 2(4 - 7x)$.

Inclose within parentheses preceded by a minus sign.

14. $-3x + 5y - 7$.
15. $4 - 2x - 5a$.
16. $-2a + 4 - 5y$.

Reduce to lowest terms without a minus sign in numerator or denominator.

17. $\frac{18}{30}$.
18. $\frac{32}{72}$.
19. $\frac{45}{21}$.
20. $\frac{2d}{7d}$.
21. $\frac{ab}{bc}$.
22. $\frac{-66}{77}$.
23. $\frac{36}{-15}$.
24. $\frac{5ch}{35h}$.
25. $\frac{21x}{14xy}$.
26. $\frac{b}{-3}$.
27. $\frac{-2}{a}$.
28. $\frac{-2}{6b}$.
29. $\frac{-c}{-5c}$.

Express the result as a fraction in lowest terms.

30. $\frac{3}{5}\cdot\frac{2}{7}$.
31. $\frac{4}{5}\cdot\frac{3}{8}$.
32. $\frac{3}{2} \div \frac{4}{5}$.
33. $\frac{11}{7} \div \frac{22}{5}$.
34. $\frac{h}{y} \div \frac{1}{5}$.
35. $\frac{ck}{6d}\cdot\frac{3}{k}$.
36. $5\cdot\frac{3}{8}$.
37. $3\cdot\frac{2}{7}$.
38. $\frac{8}{15} \div 3$.
39. $\frac{3}{5} \div 2$.
40. $5 \div \frac{3}{10}$.
41. $5d \div \frac{3d}{c}$.

42. $\dfrac{\frac{3}{2}}{\frac{15}{4}}$.
43. $\dfrac{\frac{4}{15}}{\frac{7}{3}}$.
44. $\dfrac{\frac{8}{9c}}{\frac{4d}{5c}}$.
45. $\dfrac{\frac{12a}{5b}}{\frac{8a}{15}}$.

46. $8\frac{1}{3} \div 75$.
47. $2\frac{3}{5} \div 4$.
48. $5 \div 3\frac{2}{7}$.
49. $2\frac{3}{4} \div 3\frac{5}{7}$.

50. $\dfrac{\frac{15}{7}}{6}$. **51.** $\dfrac{\frac{14}{5}}{10}$. **52.** $\dfrac{\frac{3h}{k}}{6}$. **53.** $\dfrac{\frac{4w}{9d}}{2w}$.

54. $\dfrac{3}{\frac{15}{7}}$. **55.** $\dfrac{21}{\frac{14}{9}}$. **56.** $\dfrac{2a}{\frac{c}{d}}$. **57.** $\dfrac{5w}{\frac{15w}{8}}$.

14. Positive integral exponents. By definition, if m is a positive integer, then $a^m = a \cdot a \cdot a \cdots a$, to m factors. We call a^m the **mth power** of the **base** a and m the **exponent** of the power. By definition, $a^1 = a$. Hence, when the exponent is 1 we shall usually omit it. We call a^2 the **square** of a and a^3 the **cube** of a. The following properties I–V of exponents are called **index laws.**

Note 1. Until we reach a later chapter, any literal number occurring in an exponent will represent a positive integer.

I. *Law of exponents for multiplication:* $a^m a^n = a^{m+n}$.

II. *Law for finding a power of a power:* $(a^m)^n = a^{mn}$.

Proof. 1. $\qquad (a^m)^n = a^m \cdot a^m \cdots a^m; \qquad$ (n factors a^m)

(By Law I) $\qquad = a^{m+m+\cdots+m}. \qquad$ (n terms m)

2. Since $(m + m + \cdots + m)$ to n terms equals mn, $(a^m)^n = a^{mn}$.

III. *Laws of exponents for division*:*

$$\frac{a^m}{a^m} = 1; \qquad \frac{a^m}{a^n} = a^{m-n} \text{ (if } m > n); \qquad \frac{a^m}{a^n} = \frac{1}{a^{n-m}} \text{ (if } n > m).$$

Proof, for the case $n > m$. By the definition of a^m and a^n,

$$\frac{a^m}{a^n} = \frac{\cancel{a} \cdot \cancel{a} \cdots \cancel{a}}{a \cdot a \cdots a \cdot \cancel{a} \cdot \cancel{a} \cdots \cancel{a}}; \qquad \begin{matrix}(m \text{ factors})\\(n \text{ factors})\end{matrix}$$

[$(n - m)$ factors a] $\qquad = \dfrac{1}{a \cdot a \cdots a} = \dfrac{1}{a^{n-m}}$.

IV. *Law for finding a power of a product:* $(ab)^n = a^n b^n$.

Proof. $\qquad (ab)^n = ab \cdot ab \cdots ab; \qquad$ (n factors ab)

(n factors a and b) $\qquad = (a \cdot a \cdots a)(b \cdot b \cdots b) = a^n b^n$.

Law IV extends to products of any number of factors. Thus,

$$(abc)^n = a^n b^n c^n.$$

* We read "$m > n$" as "m is greater than n," and "$m < n$" as "m is less than n."

V. *Law for finding a power of a quotient:* $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}.$

Proof. $\left(\dfrac{a}{b}\right)^n = \dfrac{a}{b}\cdot\dfrac{a}{b}\cdots\dfrac{a}{b};$ $\left(n \text{ factors } \dfrac{a}{b}\right)$

(n factors a)
(n factors b) $= \dfrac{a\cdot a\cdots a}{b\cdot b\cdots b} = \dfrac{a^n}{b^n}.$

Note 2. The determination of powers of numbers is called **involution.**

ILLUSTRATION 1. $(-3)^3 = (-3)\cdot(-3)\cdot(-3) = -27.$

ILLUSTRATION 2. $\dfrac{a^5}{a^5} = 1.$ $\dfrac{a^{10}}{a^2} = a^8.$ $\dfrac{a^4}{a^7} = \dfrac{1}{a^3}.$

ILLUSTRATION 3. $\left(\dfrac{3}{2}\right)^4 = \dfrac{3^4}{2^4} = \dfrac{81}{16}.$ $\left(\dfrac{4cd^2}{3x}\right)^2 = \dfrac{(4cd^2)^2}{(3x)^2} = \dfrac{16c^2d^4}{9x^2}.$

ILLUSTRATION 4. $\dfrac{-15a^3x^5}{10ax^9} = -\dfrac{3}{2}\cdot\dfrac{a^3}{a}\cdot\dfrac{x^5}{x^9} = -\dfrac{3a^{3-1}}{2x^{9-5}} = -\dfrac{3a^2}{2x^4}.$

15. Coefficients. If an algebraic term is an explicit number, or is a product of such a number and positive integral powers of letters, we call the explicit number the *numerical coefficient* of the term. Two terms are said to be *similar* if they differ only in their numerical coefficients. A sum of similar terms can be combined into a single term.

ILLUSTRATION 1. The numerical coefficient or, for short, *the* coefficient of $-2xy^4$ is -2.

ILLUSTRATION 2. $5xy^2 - 8xy^2 = (5-8)xy^2 = -3xy^2.$

16. Integral rational expressions. An algebraic term is said to be *integral and rational* in certain literal numbers if the term does not involve them or if it is the product of positive integral powers of the letters multiplied by a factor not involving them.

ILLUSTRATION 1. The terms 5, $\frac{1}{4}a^3$, and $-16a^4b^2$ are integral and rational in a and b. The term $3/x$ is *not* integral and rational in x.

An algebraic expression is called a **monomial** if the expression has just *one* term, a **binomial** if there are just *two* terms, and a **trinomial** if there are just *three* terms. Any expression with *more than one* term is called a **polynomial.** An *integral rational polynomial* is one in which each term is integral and rational. Hereafter, unless otherwise stated, any term or polynomial to which we refer will be *integral and rational* in all literal numbers involved.

ILLUSTRATION 2. $(3x + 7ab)$ is a binomial; $(3x - y - 5)$ is a trinomial.

17. Square roots. If $R^2 = A$, we call R a *square root* of A. Every positive number A has *two* square roots, one *positive* and one *negative.* The positive square root is denoted by $+\sqrt{A}$, or simply $\sqrt{A}$, and the negative square root by $-\sqrt{A}$. We call $\sqrt{A}$ a **radical** and A its **radicand.** Unless otherwise stated, *the* square root of A will mean its *positive* square root. If x is *positive* or *zero,* then $\sqrt{x^2} = x$.

Illustration 1. $\sqrt{9} = 3$ because $3^2 = 9$. The two square roots of 9 are $\pm\sqrt{9}$, or ± 3. $\sqrt{\frac{1}{4}} = \frac{1}{2}$ because $(\frac{1}{2})^2 = \frac{1}{4}$.

Note 1. In the square of a monomial, each exponent is *even* because, in squaring, each original exponent is multiplied by 2. Thus,

$$(3x^2y^3)^2 = 9x^4y^6.$$

A monomial is called a **perfect square** if it is the square of another monomial. Hence, in a perfect square each exponent is an *even integer.*

Illustration 2. $25a^2b^4$ is a perfect square because $25a^2b^4 = (5ab^2)^2$.

To find the square root of a perfect square monomial, rewrite the literal part with each exponent *divided by* 2 and multiply by the square root of the numerical coefficient.

Illustration 3. $\sqrt{16x^4y^8} = 4x^2y^4$, because

$$(4x^2y^4)^2 = 4^2(x^2)^2(y^4)^2 = 16x^4y^8.$$

To find the square root of a fraction, *find the square root of the numerator, and of the denominator, and divide.* That is,

$$\sqrt{\frac{N}{D}} = \frac{\sqrt{N}}{\sqrt{D}}.$$

Illustration 4. $\sqrt{\frac{4}{25}} = \frac{\sqrt{4}}{\sqrt{25}} = \frac{2}{5}.$ $\sqrt{\frac{a^2}{b^2}} = \frac{\sqrt{a^2}}{\sqrt{b^2}} = \frac{a}{b}.$

Note 2. At present, we shall consider $\sqrt{A}$ only if A is a perfect square monomial, or a fraction whose numerator and denominator are perfect square monomials. All literal numbers in A will be supposed positive or zero.

EXERCISE 2

Compute by use of the definition of an exponent.

1. 2^4.	**4.** $(-6)^3$.	**7.** $(\frac{1}{4})^2$.	**10.** $(-\frac{1}{3})^2$.	**13.** $(\frac{3}{4})^2$.
2. 5^3.	**5.** $(-2)^5$.	**8.** $(\frac{1}{3})^2$.	**11.** $(-\frac{1}{2})^3$.	**14.** $(\frac{5}{4})^3$.
3. 10^5.	**6.** 10^4.	**9.** $(\frac{1}{2})^4$.	**12.** $(\frac{4}{5})^2$.	**15.** $3(2^4)$.

Perform the operations by use of the laws of exponents. Reduce each fraction to lowest terms.

16. a^5a^4. **18.** $(b^2)^4$. **20.** $(ab)^4$. **22.** xx^3x^4. **24.** $(x^2y^3)^2$.

17. z^2z^n. **19.** $(y^4)^3$. **21.** $(hk)^3$. **23.** $(a^2b)^3$. **25.** $(hx^m)^4$.

26. $\frac{x^6}{x^2}$. **28.** $\frac{a^7}{a^9}$. **30.** $\frac{x^2}{x^9}$. **32.** $\left(\frac{3}{5}\right)^3$. **34.** $\left(\frac{4}{c}\right)^3$.

27. $\frac{d^5}{d^2}$. **29.** $\frac{x^3}{x^8}$. **31.** $\frac{a^9}{a^5}$. **33.** $\left(\frac{a}{4}\right)^2$. **35.** $\left(\frac{3}{a}\right)^4$.

36. $(3a)^3$. **37.** $(5y)^2$. **38.** $(-xy^2)^3$. **39.** $(-2h^3)^3$.

40. $\left(\frac{2c}{d}\right)^2$. **41.** $\left(\frac{3a}{4b^2}\right)^2$. **42.** $\left(\frac{2d}{3b^2}\right)^3$. **43.** $\left(\frac{c^2x}{5a}\right)^3$.

44. $2x^2(5x^4y)$. **46.** $-ax^4(-ax^2)$. **48.** $3a^m(5a^h)^2$.

45. $4ay(3a^2y^3)^2$. **47.** $-4bc(3b^4c)$. **49.** $2a^3b(3a^hb^2)$.

50. $\frac{a^4y^8}{a^2y^5}$. **52.** $\frac{4xy^3}{6xy^2}$. **54.** $\frac{-3w^9}{15w^7}$. **56.** $\frac{-33ax^5}{-3a^3x^2}$. **58.** $\frac{50x^5y^4}{-15x^6y}$.

51. $\frac{9w^4x^3}{3w^2x^2}$. **53.** $\frac{2a^6}{8a^9y}$. **55.** $\frac{-30x^4y^5}{6x^2y^7}$. **57.** $\frac{-32xy^3}{-6x^2y}$. **59.** $\frac{-63x^7w^2}{28x^4w^6}$.

Find each square root and check by squaring the result.

60. $\sqrt{9}$. **64.** $\sqrt{x^4}$. **68.** $\sqrt{\frac{1}{9}}$. **72.** $\sqrt{49a^2}$.

61. $\sqrt{100}$. **65.** $\sqrt{y^6}$. **69.** $\sqrt{\frac{1}{25}}$. **73.** $\sqrt{4x^8}$.

62. $\sqrt{\frac{1}{36}}$. **66.** $\sqrt{9y^4}$. **70.** $\sqrt{\frac{25}{9}}$. **74.** $\sqrt{49a^2b^6}$.

63. $\sqrt{\frac{4}{9}}$. **67.** $\sqrt{4z^{12}}$. **71.** $\sqrt{a^2}$. **75.** $\sqrt{64a^4y^6}$.

76. $\sqrt{\frac{9}{a^2}}$. **80.** $\sqrt{\frac{a^2}{w^4}}$. **84.** $\sqrt{\frac{49}{4y^6}}$. **88.** $\sqrt{\frac{9a^2b^4}{c^6w^{10}}}$.

77. $\sqrt{\frac{25}{x^2}}$. **81.** $\sqrt{\frac{z^4}{y^8}}$. **85.** $\sqrt{\frac{9z^4}{4x^6}}$. **89.** $\sqrt{\frac{9x^6}{4y^4z^2}}$.

78. $\sqrt{\frac{16}{w^4}}$. **82.** $\sqrt{\frac{4x^2}{25}}$. **86.** $\sqrt{\frac{16x^6}{9y^4z^6}}$. **90.** $\sqrt{\frac{121a^2}{9b^4z^6}}$.

79. $\sqrt{\frac{y^6}{49}}$. **83.** $\sqrt{\frac{25}{9x^2}}$. **87.** $\sqrt{\frac{81a^2}{y^4z^2}}$. **91.** $\sqrt{\frac{100x^2y^4}{49w^6}}$.

18. To multiply or divide a polynomial by a monomial, we perform the operation on each term of the polynomial and combine the results.

ILLUSTRATION 1.

$$\frac{-3x^4+6x^2-x^3}{-2x^4}=\frac{-3x^4}{-2x^4}+\frac{6x^2}{-2x^4}-\frac{x^3}{-2x^4}$$
$$=\frac{3}{2}-\frac{3}{x^2}+\frac{1}{2x}.$$

19. To multiply or divide one polynomial by another, if many terms are involved it is convenient to arrange them in ascending (or descending) powers of one letter before performing the operation.

ILLUSTRATION 1. To multiply $(x^2 + 3x^3 - x - 2)(2x + 3)$:

$$\begin{array}{lrrrrrl}
 & 3x^3 + & x^2 - & x & - 2 & & \\
(\textit{multiply}) & & & 2x & + 3 & & \\
\hline
 6x^4 + & 2x^3 - & 2x^2 - & 4x & & & \text{(Multiplying by } 2x) \\
 & 9x^3 + & 3x^2 - & 3x & - 6 & & \text{(Multiplying by 3)} \\
\hline
(\textit{add})\ 6x^4 + & 11x^3 + & x^2 - & 7x & - 6 & = & \textit{product.}
\end{array}$$

Note 1. After any step in a division, the partial quotient so far obtained and the corresponding remainder satisfy the equation

$$\frac{\textbf{dividend}}{\textbf{divisor}} = \textbf{quotient} + \frac{\textbf{remainder}}{\textbf{divisor}}. \tag{1}$$

Or, $$\textbf{dividend} = (\textbf{quotient})(\textbf{divisor}) + \textbf{remainder}. \tag{2}$$

EXAMPLE 1. Find $(6a^4 + a^2 - a^3 + 15) \div (2a^2 - 3a + 5)$.

SOLUTION. 1. Since $[(6a^4) \div (2a^2)] = 3a^2$, we subtract

$3a^2(2a^2 - 3a + 5)$, or $(6a^4 - 9a^3 + 15a^2)$, etc.

$$\begin{array}{rl}
 & 3a^2 + 4a - 1 = \textit{quotient} \\
(\textit{divisor})\ 2a^2 - 3a + 5 & \overline{)\,6a^4 - a^3 + a^2 \qquad + 15} \\
(\textit{subtract}) & 6a^4 - 9a^3 + 15a^2 \\
\hline
 & 8a^3 - 14a^2 \\
 & 8a^3 - 12a^2 + 20a \\
\hline
 & -2a^2 - 20a + 15 \\
 & -2a^2 + 3a - 5 \\
\hline
\textit{remainder} = & -23a + 20
\end{array}$$

2. Hence, $\dfrac{6a^4 + a^2 - a^3 + 15}{2a^2 - 3a + 5} = 3a^2 + 4a - 1 + \dfrac{20 - 23a}{2a^2 - 3a + 5}.$

20. Special products. We usually dispense with long-hand multiplication in finding the product of two binomials and perform the details mentally. Products of the following types occur frequently.

I. $\quad a(x + y) = ax + ay.$

II. $\quad (x + y)(x - y) = x^2 - y^2.$

III. $\quad (a + b)^2 = a^2 + 2ab + b^2.$

IV. $\quad (a - b)^2 = a^2 - 2ab + b^2.$

V. $\quad (x + a)(x + b) = x^2 + (ax + bx) + ab.$

VI. $\quad (ax + b)(cx + d) = acx^2 + (adx + bcx) + bd.$

Note 1. The terms $ax + bx$ in Type V, and $adx + bcx$ in Type VI, can be remembered as the sum of the *cross products.*

ILLUSTRATION 1. Type II states that *the product of the sum and the difference of two numbers is the difference of their squares.*

ILLUSTRATION 2. Type III states that *the square of the sum of two numbers equals the square of the first, plus twice the product of the numbers, plus the square of the second number.*

ILLUSTRATION 3. $(c - 2d)(c + 2d) = c^2 - 4d^2.$ (Type II)

$$(5c - 2h)^2 = (5c)^2 - 2(5c)(2h) + (2h)^2 \qquad \text{(Type IV)}$$
$$= 25c^2 - 20ch + 4h^2.$$

$$(4x + 5)(-2x + 3) = -8x^2 + (12x - 10x) + 15 \qquad \text{(Type VI)}$$
$$= -8x^2 + 2x + 15.$$

EXERCISE 3

Multiply and collect similar terms.

1. $2x(3x - 7)$. **2.** $-w(a - bw)$. **3.** $5y(3 - 6x - 2y^2)$.

4. $(y - 2)(y^2 + 2y + 4)$. **7.** $(a^2 - 3a + 2)(5a^2 - 2a - 3)$.

5. $(4a^2 - 2a + 1)(2a + 1)$. **8.** $(5 + 3x^2 - 4x)(5x^2 - 3 + x)$.

6. $(x^2 + 3x + 4)(x^2 - 2x + 3)$. **9.** $(y^3 + y + 2y^2 - 1)(y^2 - 2y + 3)$.

Express as a sum of fractions in lowest terms.

10. $(y^2 - my - y^3) \div (-y)$. **11.** $(21a^3b^2 - 42ab^3) \div (-7ab^2)$.

12. $\dfrac{3 - 5xy - 4x^3y^3}{20x^2y}$. **13.** $\dfrac{4x^3 - 5x^2y - 16y^2}{-2x^2y^2}$.

Perform the division. Summarize the result as in Example 1 of Section 19.

14. $(y^2 + 7y - 18) \div (y + 9)$. **15.** $(a + 6a^2 + 3) \div (2a - 1)$.

16. $(5a - 2a^2 + 3a^3 - 26) \div (a - 2)$.

17. $(x^4 - 4x^3 + 3x^2 - 4x + 12) \div (x - 3)$.

18. $(4x^3 - 9x - 8x^2 + 7) \div (2x - 3)$.

19. $(30y - 8 - 19y^2 - 15y^3) \div (3y^2 + 5y - 4)$.

20. $(8x^2 - 3 - 5x^3) \div (7x - 2 + 5x^2)$.

21. $(z^3 - 27) \div (z - 3)$. **23.** $(x^5 + 32) \div (x + 2)$.

22. $(16x^4 - 1) \div (2x - 1)$. **24.** $(8w^3 - 27) \div (2w - 3)$.

Expand by use of the type products of Section 20.

25. $5(3a - 4v)$. **28.** $(u + v)^2$. **31.** $(2h - 3k)^2$.

26. $3c(2 - 6c)$. **29.** $(a - x)^2$. **32.** $(3x + 5y)^2$.

27. $-2h(x - y)$. **30.** $(h + 5)^2$. **33.** $(-a + b)^2$.

34. $(x - 4)(x + 4)$.

35. $(2 - 3x)(2 + 3x)$.

36. $(3 + y)(2 + y)$.

37. $(2 - a)(4 - a)$.

38. $(4 + x)(3 - 2x)$.

39. $(-2 + 3x)(5 - 2x)$.

40. $(4a - 3b)(4a + 3b)$.

41. $(2x - 5y)(2x + 5y)$.

42. $(-3 - 5a)(7 + 4a)$.

43. $(3x - 4ay)(3x + 4ay)$.

44. $(1 - 2x^2)(1 + 2x^2)$.

45. $(x - \frac{1}{2})(x + \frac{1}{2})$.

46. $(2xy - 3y^2)^2$. **47.** $(x - 2x^2w)^2$. **48.** $(y + \frac{1}{4})^2$. **49.** $(\frac{1}{3} - z)^2$.

50. $[2(x - y)]^2$. **51.** $[3(a + b)]^2$. **52.** $[5(2c - d)]^2$.

53. $(c + 2d - 11a)(c + 2d + 11a)$.

SOLUTION. $[(c + 2d) - 11a][(c + 2d) + 11a]$ (Type II)

$= (c + 2d)^2 - (11a)^2 = c^2 + 4cd + 4d^2 - 121a^2.$

Employ Types II *to* VI *in expanding.*

54. $[(x + y) - 3][(x + y) + 3]$.

55. $[(c + y) - 2][(c + y) + 2]$.

56. $[(x + y) + 3]^2$.

57. $[(a - b) + 5]^2$.

58. $[2z - (3x^2 + y)]^2$.

59. $[2a + (3b - x)]^2$.

60. $(3a + 2b + c)^2$.

61. $(4a - b - c)^2$.

62. $(3x + y - 2)(3x + y + 2)$.

63. $(3a - y + 4)(3a - y - 4)$.

64. $(a + b + c + 2)(a + b - c - 2)$.

65. $(c - 2d - a - x)(c - 2d + a + x)$.

66. Expand $(x + y + z)^2$ and state the result in words.

Use the formula of Problem 66 *to expand each square.*

67. $(2a + 3b + 4c)^2$.

68. $(w - 5x + 3a)^2$.

69. $(3a - 2b + 3c)^2$.

70. $(4z^3 - 3xy^2 - 5x^3)^2$.

21. Factoring. In our discussion of factoring, unless otherwise stated, the coefficients will be *integers* in any polynomial referred to. Such an expression will be called **prime** if it has no integral rational factors except *itself*, or its *negative*, or *1*. No simple rule can be stated for determining whether or not an expression is prime.

ILLUSTRATION 1. We shall say that $(x - y)$ is prime although

$$x - y = (\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y}),$$

because these factors are not integral and rational. Other prime expressions are $(x + y)$, $(x^2 + y^2)$, $(x^2 + xy + y^2)$, and $(x^2 - xy + y^2)$.

To factor a polynomial will mean to express it as a product of positive integral powers of distinct *prime* factors.

22. Factoring by inspection. Each type formula of Section 20 becomes a formula for factoring when read *from right to left.* In particular, Types II, III, and IV are recalled as follows:

The difference of two squares:

II. $$x^2 - y^2 = (x - y)(x + y).$$

Perfect square trinomials:

III. $$a^2 + 2ab + b^2 = (a + b)^2;$$

IV. $$a^2 - 2ab + b^2 = (a - b)^2.$$

In a perfect square trinomial, we notice that

1. *two terms are perfect squares, and*

2. *the third term is plus (or minus) twice the product of the square roots of the other terms.*

ILLUSTRATION 1. By use of Type I of Section 20, we remove a common monomial factor:

$$3x^2y^3z - 4x^2y^3w - 4ax^2y^3 = x^2y^3(3z - 4w - 4a).$$

ILLUSTRATION 2. $4w^2 - 25y^4z^4 = (2w - 5y^2z^2)(2w + 5y^2z^2)$. (Type II)

ILLUSTRATION 3. $25 - 20xy + 4x^2y^2 = (5 - 2xy)^2$. (Type IV)

Certain trinomials of the form * $gx^2 + hx + k$ can be factored by a trial and error method based on the formulas of Types V and VI.

EXAMPLE 1. Factor $15x^2 + 2x - 8$.

SOLUTION. 1. We wish to find a, b, c, and d so that

$$(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd = 15x^2 + 2x - 8.$$

2. Hence, $ac = 15$, $bd = -8$, and the sum of the cross products is $2x$. After various unsatisfactory trials, we finally select $a = 3$, $c = 5$, $b = -2$, and $d = 4$, and verify that

$$15x^2 + 2x - 8 = (3x - 2)(5x + 4).$$

If one prime factor is merely the *negative* of another, we do *not* consider them as distinct prime factors; we combine their powers into a single power of one of them.

ILLUSTRATION 4. In $(-x - 2)(x + 2) = -x^2 - 4x - 4$, we notice that $(-x - 2) = -(x + 2)$. Hence, we write

$$-x^2 - 4x - 4 = -(x + 2)(x + 2) = -(x + 2)^2.$$

* If g, h, and k were chosen at random, without a common factor, the trinomial would probably be prime. Later, we shall discuss a condition which g, h, and k satisfy when and only when the trinomial is *not* prime.

23. Factoring by grouping.

ILLUSTRATION 1. $6 - 3x^2 - 8x + 4x^3 = (6 - 8x) - (3x^2 - 4x^3)$
$= 2(3 - 4x) - x^2(3 - 4x) = (3 - 4x)(2 - x^2).$

ILLUSTRATION 2. $a^2 - c^2 + b^2 - d^2 - 2ab - 2cd$
$= (a^2 - 2ab + b^2) - (c^2 + 2cd + d^2) = (a - b)^2 - (c + d)^2$
$= [(a - b) - (c + d)][(a - b) + (c + d)]$
$= (a - b - c - d)(a - b + c + d).$

EXERCISE 4

Factor. If fractions occur, leave the factors in the form which arises most naturally by use of the standard methods.

1. $ax + ay.$ **2.** $ax - xw.$ **3.** $-ax + bx.$
4. $3ab + 2a - 5a^2.$ **5.** $-4at + t^2 - ct^3.$
6. $w^2 - z^2.$ **7.** $4x^2 - y^2.$ **8.** $9x^2 - 25z^2.$
9. $4a^2 - 9b^2.$ **10.** $1 - 25x^2.$ **11.** $9z^2 - \frac{1}{4}.$
12. $\frac{1}{9} - w^2.$ **13.** $25w^2 - c^2d^2.$ **14.** $36a^2b^2 - 64x^2.$

Insert any missing term to complete a perfect square; then factor.

15. $w^2 + 12w + 36.$
16. $a^2 + 9 - 6a.$
17. $x^2 + x + \frac{1}{4}.$
18. $49x^2 + 14ax + a^2.$
19. $4x^2 + (\quad) + 9z^2.$
20. $16b^2 - (\quad) + 49x^2.$
21. $a^2b^2 - (\quad) + 9x^2.$
22. $9x^2 - (\quad) + w^2h^2.$

Factor.

23. $x^2 + 8x + 15.$
24. $x^2 + 10x + 21.$
25. $x^2 - x - 6.$
26. $x^2 + x - 30.$
27. $2x^2 + 7x + 3.$
28. $5a^2 + 12a + 7.$
29. $15a^2 + 14a - 8.$
30. $6x^2 + x - 15.$

Factor without first expanding any part of the expression.

31. $3a(x + y) - 5b(x + y).$
32. $2h(a - b) + 3(a - b).$
33. $bx + by + 2hx + 2hy.$
34. $3ac + 3bc + ad + bd.$
35. $4hx - 4bh - 5cx + 5bc.$
36. $3bw - 3bz - 4aw + 4az.$
37. $(x^3 - 2x^2) - (x - 2).$
38. $ac^2 + c^3 - ad^2 - cd^2.$
39. $bx^2 + cx^2 - by^2 - cy^2.$
40. $x^2 - (s^2 + 6s + 9).$

SOLUTION of Problem 40. The expression is the difference of two squares:

$$x^2 - (s + 3)^2 = [x - (s + 3)][x + (s + 3)]$$
$$= (x - s - 3)(x + s + 3).$$

41. $c^2 - 49(a + b)^2$.

42. $(x + y)^2 - 16(z - w)^2$.

43. $6a^2 - 11ab + 5b^2$.

44. $6a^2 - 5ab - 4b^2$.

45. $64a^2 + 9c^2 - 48ac$.

46. $9 + 25v^2w^2 - 30vw$.

47. $2r - 11hr + 15h^2r$.

48. $3x^4 - 7x^2 - 20$.

49. $5x^4 - 16x^2 + 3$.

50. $9(a + b)^2 + 12(a + b) + 4$.

51. $8a^2c - 18c^3$.

52. $x^4 - y^4$.

53. $81c^4 - 16d^4$.

54. $(n^2 - 4n + 4) - 16w^2$.

55. $4a^2 + 12a + 9 - 25x^2$.

56. $9y^2 - (4x^2 + 20x + 25)$.

57. $4h^2 - (9a^2 - 12a + 4)$.

58. $4w^2 + 20w + 25 - 81z^2$.

59. $y^2 + 2yz + z^2 - 4x^2$.

60. $9w^2 - 4a^2 - 4ab - b^2$.

61. $4a^2 - 9z^2 - 6z - 1$.

62. $4x^2 + 4xy + y^2 - 9a^2 - 12at - 4t^2$.

63. $15 + (z + w) - 6(z + w)^2$.

64. $6(x - y)^2 - 11(x - y) + 4$.

65. $(a^2 + 2a)^2 + 2(a^2 + 2a) + 1$.

66. $10(z + w)^2 + 14(z + w) - 12$.

67. $20(x + y)^2 - 22w(x + y) + 6w^2$.

68. $(1 + n)^2 - 4m(1 + n) + 4m^2$.

★69. $64a^4 - 64a^2b^2 + 25b^4$.

SOLUTION of Problem 69. 1. A perfect square involving $64a^4$ and $25b^4$ is

$$(8a^2 + 5b^2)^2 = 64a^4 + 80a^2b^2 + 25b^4.$$

2. Hence, $64a^4 - 64a^2b^2 + 25b^4$ becomes a perfect square if we add $144a^2b^2$. Therefore, we add $144a^2b^2$ and, to compensate for this, also subtract $144a^2b^2$:

$$\begin{aligned} 64a^4 - 64a^2b^2 + 25b^4 &= (64a^4 - 64a^2b^2 + 25b^4 + 144a^2b^2) - 144a^2b^2 \\ &= (64a^4 + 80a^2b^2 + 25b^4) - 144a^2b^2 = (8a^2 + 5b^2)^2 - 144a^2b^2 \\ &= (8a^2 + 5b^2 - 12ab)(8a^2 + 5b^2 + 12ab). \end{aligned}$$

★*Factor by reducing to a difference of two squares.*

70. $a^4 + a^2 + 1$.

71. $z^4 - 3z^2 + 1$.

72. $9x^4 + 11x^2 + 4$.

73. $z^4 - 6z^2 + 1$.

74. $4w^4 + 8a^2w^2 + 9a^4$.

75. $a^4 - 9a^2y^2 + 16y^4$.

76. $x^4 + 4$.

77. $w^4 + 4x^4$.

78. $625x^4 + 4w^4$.

24. The cube of a binomial. We verify the following formulas by direct multiplication on recalling that $a^3 = a \cdot a \cdot a$.

$$(x + y)^3 = \mathbf{x^3 + 3x^2y + 3xy^2 + y^3}. \qquad (1)$$

$$(x - y)^3 = \mathbf{x^3 - 3x^2y + 3xy^2 - y^3}. \qquad (2)$$

ILLUSTRATION 1. From formula 1, with $x = 2a$ and $y = b$,

$$\begin{aligned} (2a + b)^3 &= (2a)^3 + 3(2a)^2(b) + 3(2a)(b^2) + b^3 \\ &= 8a^3 + 12a^2b + 6ab^2 + b^3. \end{aligned}$$

25. Factors of the sum and the difference of two cubes.

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2). \tag{1}$$

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2). \tag{2}$$

ILLUSTRATION 1. From formula 2 with $a = 3x$ and $b = 2y$,

$$\begin{aligned} 27x^3 + 8y^3 &= (3x)^3 + (2y)^3 \\ &= (3x + 2y)[(3x)^2 - (3x)(2y) + (2y)^2] \\ &= (3x + 2y)(9x^2 - 6xy + 4y^2). \end{aligned}$$

26. The sum and the difference of two like powers. We have verified special cases of the following results, where n represents a positive integer. Any special case of the results can be checked by long division.

I. *For every value of n, $(a^n - b^n)$ has $(a - b)$ as a factor.*

ILLUSTRATION 1. $a^4 - b^4 = (a - b)(a^3 + a^2b + ab^2 + b^3)$.

II. *If n is even, $(a^n - b^n)$ has $(a + b)$ as a factor.*

ILLUSTRATION 2. $a^4 - b^4 = (a + b)(a^3 - a^2b + ab^2 - b^3)$.

III. *If n is odd, $(a^n + b^n)$ has $(a + b)$ as a factor.*

ILLUSTRATION 3. $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

$$a^7 + b^7 = (a + b)(a^6 - a^5b + a^4b^2 - a^3b^3 + a^2b^4 - ab^5 + b^6).$$

IV. *If n is even, $(a^n + b^n)$ does not have either $(a - b)$ or $(a + b)$ as a factor.*

ILLUSTRATION 4. $(a^2 + b^2)$ and $(a^4 + b^4)$ are prime. $(a^6 + b^6)$ is not prime but it does not have either $(a + b)$ or $(a - b)$ as a factor:

$$a^6 + b^6 = (a^2 + b^2)(a^4 - a^2b^2 + b^4).$$

Special cases of the following general properties were exhibited by the second factors in Illustrations 1, 2, and 3.

A. *When $(a^n - b^n)$ is divided by $(a - b)$, the coefficients in the quotient are all $+ 1$.*

B. *When $(a^n + b^n)$ or $(a^n - b^n)$ is divided by $(a + b)$, the coefficients in the quotient are alternately $+ 1$ and $- 1$.*

Factors obtained by reference to results I, II, and III are not always prime. Also, as we have seen in Illustration 4, an expression $a^n + b^n$, with n even, may be factorable although result IV is true. In order to find the prime factors of $a^n + b^n$, it is not always desirable

to use results I, II, and III immediately. Thus, to factor $a^n - b^n$ when n is even, first factor as the *difference of two squares.* To factor $a^n \pm b^n$ when n is odd, first factor as *the sum or the difference of two cubes* of powers of a and b, if possible.

ILLUSTRATION 5. $x^9 - y^9 = (x^3)^3 - (y^3)^3 = (x^3 - y^3)(x^6 + x^3y^3 + y^6)$
$= (x - y)(x^2 + xy + y^2)(x^6 + x^3y^3 + y^6).$

ILLUSTRATION 6. $x^4 - y^4 = (x^2)^2 - (y^2)^2 = (x^2 - y^2)(x^2 + y^2)$
$$= (x - y)(x + y)(x^2 + y^2). \tag{1}$$

By use of result I,

$$x^4 - y^4 = (x - y)(x^3 + x^2y + xy^2 + y^3). \tag{2}$$

Equation 1 shows that the second factor in (2) is not prime.

EXERCISE 5

Divide by long division, and check by Sections 25 and 26.

1. $\dfrac{a^3 - h^3}{a - h}$.   2. $\dfrac{x^3 + 8y^3}{x + 2y}$.   3. $\dfrac{x^4 - y^4}{x - y}$.   4. $\dfrac{x^5 + y^5}{x - y}$.

Multiply by inspection.

5. $(c + w)(c^2 - cw + w^2)$.   7. $(3a - c)(9a^2 + 3ac + c^2)$.
6. $(u - v)(u^2 + uv + v^2)$.   8. $(1 - 3x)(1 + 3x + 9x^2)$.

Factor.

9. $d^3 - y^3$.   11. $1 - v^3$.   13. $z^3 + 1000$.   15. $216x^3 - y^3z^3$.
10. $h^3 + z^3$.   12. $27 - u^3$.   14. $8x^3 - 125y^3$.   16. $x^3 - 343y^3$.

Expand each cube by use of the formulas of Section 24.

17. $(c + d)^3$.   19. $(5 - y)^3$.   21. $(y - 3x)^3$.   23. $(c + 3b^2)^3$.
18. $(2 + y)^3$.   20. $(2x + w)^3$.   22. $(a - b^2)^3$.   24. $(c - 6z^3)^3$.

Find each result without using long division, by use of properties A *and* B *of Section 26, and check by multiplication.*

25. $\dfrac{a^5 + y^5}{a + y}$.   26. $\dfrac{u^8 - v^8}{u - v}$.   27. $\dfrac{x^5 - 1}{x - 1}$.   28. $\dfrac{z^5 - 32x^{10}}{z - 2x^2}$.

29. $(a^8 - 16b^4) \div (a^2 - 2b)$.   30. $(243x^5 - 1) \div (3x - 1)$.

Factor each expression which is not prime.

31. $a^5 - c^5$.   33. $y^4 - 81$.   35. $h^9 + k^9$.   37. $a^3 - 27x^6$.
32. $a^4 - w^4$.   34. $32 + x^5$.   36. $a^6 - 64$.   38. $32x^{5k} - w^5$.
39. $4x^4 + 1$.   41. $x^{16} + y^{16}$.   43. $16x^4 - y^8$.
40. $16x^4 + y^4$.   42. $4w^4x^4 + 81z^4$.   44. $u^{3h} + v^3$.

27. The degree of an integral rational term in certain letters is defined as the sum of the exponents with which these letters appear in the term. The degree of an integral rational polynomial is defined as the degree of its term of *highest* degree.

ILLUSTRATION 1. x^4yz^2 is of the 4th degree in x, 1st degree in y, and 7th degree in x, y, and z. $(x^3 + x^2 - 7)$ is of the 3d degree in x.

28. The lowest common multiple (L.C.M.) of two or more integral rational expressions, with integral coefficients, is the expression of *lowest* degree, with *smallest* integral coefficients, which has each given expression as a factor. To find the L.C.M., factor the expressions and form the product of *all their different prime factors*, giving each factor the *highest* exponent with which it appears in any expression. Two results for a L.C.M. which differ *only in sign* will be considered essentially identical.

ILLUSTRATION 1. The L.C.M. of

$$2(3 - x)(3 + x), \quad 4(x - 3)(x - 1), \quad \text{and} \quad 3(x - 3)^2$$

is $4 \cdot 3(x - 3)^2(x + 3)(x - 1)$. We did not consider $(3 - x)$ and $(x - 3)$ as distinct factors because $3 - x = -(x - 3)$.

Note 1. The **highest common factor (H.C.F.)** of two or more integral rational expressions is the expression of *highest* degree, with *largest* integral coefficients, which is a factor of each of the given expressions. Thus, the H.C.F. of $6x^2y^3$ and $4xy^4$ is $2xy^3$. We shall not find it essential to use the H.C.F. terminology.

29. To reduce a fraction to lowest terms, factor the numerator and denominator and then divide both of them by all their common factors.

ILLUSTRATION 1.
$$\frac{x^2 - 9}{12 + 2x - 2x^2} = \frac{(x - 3)(x + 3)}{2(3 - x)(2 + x)}$$

$$= -\frac{\cancel{(x - 3)}(x + 3)}{2\cancel{(x - 3)}(2 + x)} = -\frac{x + 3}{2(x + 2)}. \qquad \text{(Divide out } x - 3\text{)}$$

In the preceding line, we obtained $(x - 3)$ in the denominator by multiplying $(3 - x)$ by -1, and hence it was necessary to change the sign before the fraction to keep its value unaltered.

30. Addition of fractions. The *sum* (or the *difference*) of two fractions with a common denominator is a fraction whose denominator is the *common denominator* and whose numerator is the *sum* (or the *difference*) of the numerators of the given fractions.

ILLUSTRATION 1. $\frac{3}{4d} - \frac{a-b}{4d} + \frac{c}{4d} = \frac{3-(a-b)+c}{4d}$

$$= \frac{3-a+b+c}{4d}.$$

The **lowest common denominator** (**L.C.D.**) of two or more fractions is the L.C.M. of their denominators. To express a sum of fractions as a single fraction,

1. *factor the denominators and find the* L.C.D.;
2. *for each fraction, divide the* L.C.D. *by the denominator and then multiply both numerator and denominator by the resulting quotient, to express the fraction as an equal one having the* L.C.D.;
3. *combine the new numerators just obtained, and divide by the* L.C.D.

Note 1. Recall that any polynomial may be written as a fraction whose denominator is 1. Thus, $x = x/1$.

EXAMPLE 1. Express as a single fraction: $\frac{4x}{x^2-9} - \frac{3x}{x^2-5x+6}$.

SOLUTION. 1. The denominators are

$$x^2 - 9 = (x-3)(x+3), \text{ and } x^2 - 5x + 6 = (x-3)(x-2).$$

Hence, the L.C.D. is $(x-3)(x+3)(x-2)$.

2. For the first fraction, $[(\text{L.C.D.}) \div (x^2 - 9)] = x - 2$.

For the second fraction, $[(\text{L.C.D.}) \div (x^2 - 5x + 6)] = x + 3$.

3. We multiply both numerator and denominator by $x - 2$ in the first fraction and by $x + 3$ in the second:

$$\frac{4x}{x^2-9} - \frac{3x}{x^2-5x+6}$$
$$= \frac{4x(x-2)}{(x-3)(x+3)(x-2)} - \frac{3x(x+3)}{(x-3)(x-2)(x+3)}$$
$$= \frac{4x(x-2) - 3x(x+3)}{(x-3)(x+3)(x-2)} = \frac{x^2 - 17x}{(x-3)(x+3)(x-2)}$$

EXERCISE 6

Write in lowest terms with no minus signs in the fraction.

1. $\frac{95a^2}{19a}$. 2. $\frac{25x^3y^4}{50x^2y^6}$. 3. $\frac{-3}{a}$. 4. $\frac{-5(-a)}{-x-b}$. 5. $\frac{2a+2b}{-3a-3b}$.

Reduce to lowest terms.

6. $\frac{ax(x+y)}{cx(x+y)}$. 7. $\frac{c^2d(x-y)}{2c(x-y)}$. 8. $\frac{ax-cx}{a^2-c^2}$.

9. $\dfrac{x^2-4}{2x-4}$.

10. $\dfrac{4a^2-9b^2}{2ax-3bx}$.

11. $\dfrac{4x^2-4y^2}{cx+cy}$.

12. $\dfrac{m^2-m-42}{m^2-3m-28}$.

13. $\dfrac{m^2+2m-15}{m^2+m-20}$.

14. $\dfrac{3x^2+13xy-10y^2}{9x^2-4y^2}$.

15. $\dfrac{3x^2-7ax+4a^2}{3x^2+2ax-8a^2}$.

16. $\dfrac{2x^2+2ax-12a^2}{3x^2+12ax+9a^2}$.

17. $\dfrac{a^2-4ax-5x^2}{2a^2-9ax-5x^2}$.

18. $\dfrac{a^3-b^3}{2a^2-2b^2}$.

19. $\dfrac{ax+ay}{3x^3+3y^3}$.

20. $\dfrac{27x^3-8y^3}{3x^2+xy-2y^2}$.

Combine into a single fraction in lowest terms.

21. $\dfrac{3x}{2a}-\dfrac{5y}{3b}$.

22. $\dfrac{4}{3a^2}-\dfrac{5y-1}{5ab}$.

23. $\dfrac{3a}{4c}-\dfrac{5-6c}{a}+1$.

24. $3a-\dfrac{3-7a}{2a-3}+1$.

25. $\dfrac{2x-1}{2x+3}-\dfrac{x+2}{2-3x}$.

26. $\dfrac{3x}{4x+1}-\dfrac{2x-5}{x+3}$.

27. $\dfrac{a-4}{a-2}+\dfrac{2-11a}{2-a}$.

28. $\dfrac{2a-n}{2a-2n}+\dfrac{3a-4n}{6n-6a}$.

29. $\dfrac{5x}{x+4}-\dfrac{4x^2+2x-1}{x^2+x-12}$.

30. $\dfrac{2x+1}{x^2+4x-60}-\dfrac{2}{x-6}$.

31. $\dfrac{1}{n-1}-\dfrac{n+6}{n^2+3n-4}$.

32. $1-\dfrac{6}{y}+\dfrac{1}{y^2-4y}$.

33. $\dfrac{a}{4x-x^2}+\dfrac{2a}{3x^2-48}$.

34. $\dfrac{4}{y}+1-\dfrac{y-1}{y^2+5y}$.

35. $\dfrac{x^2+5}{8x^3-27}-\dfrac{3x+2}{2x-3}$.

31. Multiplication and division of fractions with integral rational terms. To form the product of two fractions, or to divide one fraction by another, first factor their numerators and denominators.

ILLUSTRATION 1. $\dfrac{2x^2+7x-15}{2x^2-3x-14}\cdot\dfrac{2x^2-19x+42}{8x-12}$

$$=\frac{(2x-3)(x+5)}{(2x-7)(x+2)}\cdot\frac{(2x-7)(x-6)}{4(2x-3)}=\frac{(x+5)(x-6)}{4(x+2)}.$$

ILLUSTRATION 2. $\dfrac{\dfrac{xy^2-y^3}{x^3+x^2y}}{\dfrac{x^2-2xy+y^2}{x^2-xy-2y^2}}=\dfrac{xy^2-y^3}{x^3+x^2y}\cdot\dfrac{x^2-xy-2y^2}{x^2-2xy+y^2}$

$$=\frac{y^2(x-y)}{x^2(x+y)}\cdot\frac{(x-2y)(x+y)}{(x-y)^2}=\frac{y^2(x-2y)}{x^2(x-y)}.$$

A **simple fraction** is one without any fraction in its numerator or denominator. A **complex fraction** is one in which one or more fractions appear in the numerator and denominator. A **mixed expression** is one consisting of an integral rational part and one or more fractions. It is frequently desirable to combine a mixed expression into a single fraction before performing other operations.

32. To reduce a complex fraction to a simple fraction, we may first reduce the numerator and the denominator to *simple fractions* and then form their quotient.

ILLUSTRATION 1. $$\frac{1+\frac{3}{5}}{2-\frac{4}{3}} = \frac{\frac{5+3}{5}}{\frac{6-4}{3}} = \frac{\frac{8}{5}}{\frac{2}{3}} = \frac{8}{5}\cdot\frac{3}{2} = \frac{12}{5}.$$

Sometimes it may be convenient to reduce a complex fraction to a simple fraction by the single operation of *multiplying both numerator and denominator of the complex fraction by the* L.C.D. *of all simple fractions involved.*

ILLUSTRATION 2. To reduce the given complex fraction of Illustration 1 to a simple fraction, multiply both numerator and denominator by 15:

$$\frac{1+\frac{3}{5}}{2-\frac{4}{3}} = \frac{15+9}{30-20} = \frac{24}{10} = \frac{12}{5}.$$

ILLUSTRATION 3. $$\frac{\frac{a-a^2}{a^2-1}}{\frac{a}{a+1}-a} = \frac{\frac{a-a^2}{a^2-1}}{\frac{a-a(a+1)}{a+1}} = -\frac{\frac{a-a^2}{a^2-1}}{\frac{a^2}{a+1}}$$

$$= -\frac{a(1-a)}{(a-1)(a+1)}\cdot\frac{a+1}{a^2} = -\frac{1-a}{a(a-1)} = \frac{1}{a}.$$

DEFINITION I. *The* **reciprocal** *of a number H is* $\frac{1}{H}$.

ILLUSTRATION 4. The reciprocal of $\frac{3}{4}$ is $\frac{1}{\frac{3}{4}} = \frac{1}{1}\cdot\frac{4}{3} = \frac{4}{3}$.

The reciprocal of $\frac{a}{b}$ is $\frac{1}{\frac{a}{b}} = \frac{1}{1}\cdot\frac{b}{a} = \frac{b}{a}$. Thus, the reciprocal of a fraction is the fraction *inverted.*

EXERCISE 7

Perform the indicated operations and express the result as a simple fraction in lowest terms.

1. $\dfrac{3}{5} \cdot \dfrac{c}{6d}$.
2. $\dfrac{3}{5x} \div \dfrac{6}{10x^2}$.
3. $\dfrac{2cx^2}{d^3} \div 10cx$.
4. $12c^3 \div \dfrac{3cx}{d^2}$.
5. $\dfrac{hx - hy}{ab - ac} \cdot \dfrac{bw - cw}{3x - 3y}$.
6. $(h^2 - x^2) \cdot \dfrac{5w}{ch - cx}$.
7. $\dfrac{x - 1}{x^2 - 4x} \cdot (x^2 - 16)$.
8. $\dfrac{9y^2 - 1}{y^2 - 16} \cdot \dfrac{y^2 + 4y}{6y - 2}$.
9. $\dfrac{3c - bc}{5w - aw} \div \dfrac{3a - ab}{5k - ak}$.
10. $\dfrac{2x - 2y}{6x + 3y} \div \dfrac{(x - y)^2}{4x^2 - y^2}$.
11. $\dfrac{h^2 - 9}{3x - 3y} \div \dfrac{h^2 - 6h + 9}{y^2 - x^2}$.
12. $(5x - 3x^2) \div \dfrac{25 - 9x^2}{x + 3}$.
13. $\dfrac{16x + 4}{5x + 5} \cdot \dfrac{x^2 - 1}{6x - 6} \cdot \dfrac{x^2 - 2x + 1}{16x^2 - 1}$.
14. $\dfrac{2a^2 - 5ab}{c - 3d} \div (4a^2 - 25b^2)$.
15. $(a^2 + 3ax - 4x^2) \div \dfrac{ab + 4bx}{5a - x}$.
16. $\dfrac{15 - \frac{2}{6}}{\frac{1}{3} + \frac{1}{5}}$.
17. $\dfrac{1 + \frac{9}{11}}{7}$.
18. $\dfrac{3y - \frac{2}{5}}{4y^2 + \frac{3}{7}}$.
19. $\dfrac{x^2 - \frac{1}{9}}{y^2 - \frac{1}{4}}$.
20. $\dfrac{1 + \dfrac{1}{3a}}{1 - \dfrac{1}{9a^2}}$.
21. $\dfrac{\dfrac{a - a^2}{a^2 - 1}}{\dfrac{a}{a + 1} - a}$.
22. $\dfrac{c - \dfrac{2cd}{c + 2d}}{c + \dfrac{2cd}{c + 2d}}$.
23. $\dfrac{\dfrac{2a}{a + b} - 1}{\dfrac{a}{a + b} - 1}$.
24. $\dfrac{\dfrac{2}{x} - \dfrac{3}{y}}{\dfrac{4}{x^2} - \dfrac{9}{y^2}}$.
25. $\dfrac{3x - \dfrac{2}{y}}{9y^2 - \dfrac{4}{x^2}}$.
26. $\dfrac{\dfrac{a^3}{4} - \dfrac{1}{4a}}{\dfrac{a}{2} - \dfrac{1}{2a}}$.
27. $\dfrac{a^3b^3 - \dfrac{1}{ab}}{1 - \dfrac{1}{a^2b^2}}$.
28. $\dfrac{2x + \dfrac{1}{5y}}{10x^2 - \dfrac{1}{10y^2}}$.
29. $\dfrac{2x + 3a}{\dfrac{4x^2}{9} - a^2}$.
30. $\dfrac{5ac + 3bc}{5 + \dfrac{3b}{a}}$.
31. $\dfrac{9a^2 - 4}{1 + \dfrac{2}{3a}}$.
32. $\left(\dfrac{a}{9b} - \dfrac{b}{a}\right)\left(\dfrac{6a^3b^4}{a + 3b}\right)$.
33. $\left(\dfrac{c^2}{9d} - \dfrac{3d^2}{c}\right)\left(\dfrac{36c^3d^2}{c - 3d}\right)$.
34. $\dfrac{5c + \dfrac{8d^3}{25c^2}}{5ac + 2ad}$.
35. $\dfrac{\dfrac{6}{b^2} - \dfrac{1}{ab} - \dfrac{12}{a^2}}{3a + 4b}$.
36. $\dfrac{\dfrac{nx + vx - an - av}{n^2 - v^2}}{cx - ac}$.

Find the reciprocal of each expression, and express the result as a fraction in lowest terms.

37. 75. **38.** $\frac{8}{3}$. **39.** $-12\frac{2}{3}$. **40.** $(c - 2d)$. **41.** $(5d - 3a)$.

42. $\dfrac{4x - 7}{3x + 3}$. **43.** $\dfrac{5a - 3b}{6a + 2c}$. **44.** $\left(4 - \dfrac{5}{x}\right)$.

Reduce to a simple fraction in lowest terms.

45. $(6x^2 - 5xy - 15y^2) \div \left(4 - \dfrac{x + 7y}{x}\right)$.

46. $(a^4 - b^4) \div \left(\dfrac{3}{b^4} + \dfrac{1}{a^2b^2} - \dfrac{2}{a^4}\right)$.

47. $\left(\dfrac{a - 2}{a} - \dfrac{2}{a + 3}\right)\left(\dfrac{2}{a + 2} - \dfrac{3}{3 - a}\right)$.

48. $\left(2 - \dfrac{x^2 + 3x - 21}{x^2 + 2x - 8}\right) \div \left(\dfrac{x + 2}{x - 2} + \dfrac{x - 3}{x + 4}\right)$.

49. $\dfrac{1 + \dfrac{1}{x}}{x - \dfrac{1}{2x + \dfrac{x + 1}{x}}}$.

51. $\dfrac{3 - \dfrac{2a}{5a - 1}}{4a - \dfrac{2a}{1 - \dfrac{a}{1 + 2a}}}$.

50. $\dfrac{2a^2}{5a - \dfrac{4a - 1}{1 + \dfrac{2a + 5}{3a - 2}}}$.

52. $\dfrac{1 + \dfrac{2a^2}{5a - 3}}{2a + \dfrac{\dfrac{4a - 6a^2}{a - 1} - 3}{3 + \dfrac{1}{a - 1}}}$.

33. Equations. An *equation* is a statement that two number expressions are equal. The two expressions are called the *sides* or *members* of the equation. An **identity, or identical equation,** is one in which the members are equal for all permissible values of the letters involved. A **conditional equation** is one whose members are *not* equal for all permissible values of the letters.

ILLUSTRATION 1. $(a - b)^2 = a^2 - 2ab + b^2$ is an identity. $x - 2 = 0$ is a conditional equation whose members are equal only when $x = 2$.

The word *equation* by itself will be used in referring to both identities and conditional equations, except where such usage would cause confusion. Usually, however, the word *equation* refers to a conditional equation. At times, to emphasize that some equation is an identity, we use "$\equiv$" instead of "$=$" between the members.

A conditional equation may be thought of as presenting a question: the equation asks for the values which certain letters, called **unknowns,** should have in order to make the two members equal. Some letters in an equation may represent known numbers.

ILLUSTRATION 2. $x^2 + 3x - 5 = 0$ is an equation in the unknown x.

34. Solution of an equation. An equation is said to be *satisfied* by a set of values of the unknowns if the equation becomes an identity when these values are substituted for the unknowns. A **solution** of an equation is a set of values of the unknowns which satisfy the equation. A solution of an equation in a single unknown is also called a **root** of the equation. *To solve* an equation in a single unknown means to find all the solutions of the equation.

ILLUSTRATION 1. 4 is a root of the equation $2x - 3 = 5$ because, if $x = 4$, the equation becomes $2(4) - 3 = 5$, which is true.

35. Equivalent equations. Two equations are *equivalent* if they have the same solutions. Each of the following operations on an equation yields an equivalent equation.

A. *Addition of the same number to both members.*

B. *Subtraction of the same number from both members.*

C. *Multiplication (or division) of both members by the same number, provided that it is not zero and does not involve the unknowns.*

Mechanical processes grow out of operations A, B, and C.

A term appearing on both sides of an equation can be **canceled,** *by subtracting the term from both sides.*

ILLUSTRATION 1.
$$x + a = 5 + a.$$
Subtract a from both sides:
$$x = 5.$$

A term can be **transposed** *from one member to the other with the sign of the term changed, by subtracting it from both members.*

ILLUSTRATION 2.
$$x + a - 5 = 7.$$
Subtract $(a - 5)$ from both sides:
$$x = 7 - a + 5.$$

The signs of all terms on both sides may be changed, by multiplying both sides by -1.

ILLUSTRATION 3.
$$3x - b = 5 - ax.$$
Multiply both sides by -1:
$$-3x + b = -5 + ax.$$

Note 1. The following absurd result that $2 = 1$ illustrates the contradictions that arise if division by zero occurs.

1. Suppose that	$y = b.$
2. Multiply by y:	$y^2 = by.$
3. Subtract b^2:	$y^2 - b^2 = by - b^2.$
4. Factor:	$(y - b)(y + b) = b(y - b).$
5. Divide by $(y - b)$:	$y + b = b.$
6. Since $y = b$ (Step 1),	$b + b = b$, or $2b = b.$
7. On dividing both sides by b, we obtain	$2 = 1.$

Discussion. In Step 5 we divided by zero, because $y - b = 0$ if $y = b$. Hence, Steps 5, 6, and 7 are not valid, because division by zero is not allowed.

36. Linear equations. An **integral rational equation** is one where each member is an integral rational polynomial in the unknowns. A *linear* equation, or one of the *first degree*, is an integral rational equation where the terms in the unknowns are of the first degree.

Method I. *To solve a linear equation in one unknown:*

1. *Clear the equation of fractions, if present, by multiplying both sides of the equation by the* L.C.D. *of the fractions. Transpose all terms involving the unknown to one member and all other terms to the other member.*

2. *Combine terms in the unknown, exhibiting it as a factor, and divide both sides by the coefficient of the unknown.*

In the case of a linear equation in a single unknown x, if x remains in the equation after Step 1 of the standard method of solution is applied, the equation is then of the form $cx = b$, where $c \neq 0$, and b and c do not involve x. On dividing both sides of $cx = b$ by c we obtain $x = b/c$. That is, a linear equation in a single unknown has just *one root.*

Note 1. In directions for solving an equation, in this book, the words *add, subtract, multiply,* and *divide* will mean to perform the operation on *both members of the preceding equation.*

Example 1. Solve: $$\frac{27}{z-5} - \frac{8}{z+2} = \frac{18}{z^2 - 3z - 10}.$$

Solution. The L.C.D. is $(z - 5)(z + 2)$. On multiplying both sides by this L.C.D. we obtain

$$27(z + 2) - 8(z - 5) = 18; \quad z = -4.$$

The student should test the root -4 by substitution in the original equation.

37. Operations which may not yield equivalent equations.

A. *If both members of an equation are divided by an expression involving the unknowns, the new equation may have fewer roots than the original equation.*

Illustration 1. By substitution, we verify that $x = 1$ and $x = 2$ are roots of $x^2 - 3x + 2 = 0$. On dividing both sides by $(x - 2)$ we obtain

$$\frac{x^2 - 3x + 2}{x - 2} = 0,$$

$$\frac{(x - 2)(x - 1)}{x - 2} = 0, \quad \text{or} \quad x - 1 = 0.$$

The final equation has *just one* root, $x = 1$. The root $x = 2$ was lost by the division.

In solving equations, we usually avoid operations of Type A in order that roots may not be lost.

B. *If both members of an equation are multiplied by an expression involving the unknowns, the new equation thus obtained may have more solutions than the original equation.*

Illustration 2. The equation $x - 3 = 0$ has *just one* root, $x = 3$. If both sides of $x - 3 = 0$ are multiplied by $(x + 2)$ we obtain

$$(x + 2)(x - 3) = 0, \quad \text{or} \quad x^2 - x - 6 = 0.$$

By substitution, we verify that this equation has *two* roots, $x = 3$ and $x = -2$. The root -2 was introduced by the multiplication.

A value of the unknown, such as $x = -2$ in Illustration 2, which satisfies a derived equation but does *not* satisfy the original equation, is called an **extraneous root.**

Whenever an operation of Type B is employed, *test all values obtained by substituting them in the original equation, to reject extraneous roots, if any.*

Example 1. Solve the following equation:

$$\frac{x}{x^2 - 1} - \frac{1}{x^2 - 1} + \frac{2}{x + 1} = 0.$$

Solution. The L.C.D. is $x^2 - 1$; multiply both sides by $x^2 - 1$:

$$x - 1 + 2(x - 1) = 0; \quad 3x = 3; \quad \text{or} \quad x = 1.$$

Test. Since $x = 1$ makes $x^2 - 1 = 0$ in the denominators of the given equation, 1 *cannot be accepted as a root* because division by zero is not admissible. Hence, 1 is an extraneous root and therefore the given equation has *no root.*

EXERCISE 8

Solve each equation for x, or y, or z, whichever appears.

1. $2y - 4 = 1 - 4y$.
2. $2(4 - x) = 8 - 3x$.
3. $5 - 5y = 5 - 4y$.
4. $8y + 3 = 5y - 4$.
5. $7 - x = 2(1 - 4x)$.
6. $5y + \frac{1}{4} = 3y - \frac{1}{2}$.
7. $\frac{2}{3} - 3y = -5y - \frac{2}{15}$.
8. $5x - \frac{3}{4} = 3x + \frac{17}{12}$.
9. $3x - .55 = .33 - 5x$.
10. $.26 - z = .98 - 3z$.

11. $\dfrac{3x - 2}{4} + \dfrac{10x - 8}{12} = \dfrac{13}{4} + \dfrac{x - 2}{3}$.

12. $\dfrac{x - 5}{3} + \dfrac{11x - 3}{15} = 3 + \dfrac{2x - 1}{5}$.

13. $\dfrac{5 - 2x}{3} = \dfrac{25}{12} - \dfrac{5x - 3}{4}$.

14. $\dfrac{3 - x}{6} = \dfrac{5}{6} - \dfrac{x - 2}{2}$.

15. $\dfrac{3 - 2x}{3} = \dfrac{9}{5} - \dfrac{x - 3}{5}$.

16. $4 + \dfrac{2x + 7}{5} = \dfrac{7}{10} - \dfrac{3x - 5}{4}$.

17. $.21x - .46 = .79 + .96x$.
18. $.26x - .2 = .53x - .38$.
19. $.19x - .358 = .032 - .07x$.
20. $.421 - .03x = .07 - 3x$.
21. $cx - 5a = 3h$.
22. $ay - by = 5$.
23. $3x = ax + 2b$.
24. $7x - d = 5ax + 8$.
25. $ax - 3ax = 5c - bx$.
26. $3bx - 2ax = 9b^2 - 4a^2$.
27. $4az - a^2 = 4z - 1$.
28. $mnx - a = anx - m$.
29. $x + a(x + b) = ax + b$.
30. $x^2 - 3n^2 = (3n - x)^2$.
31. $acx + adx + d^2 = c^2 - bcx - bdx$.

Solve for y in terms of x, and then for x in terms of y.

32. $3x - 2y + 5 = 0$.
33. $2xy - 3x + 5y - 7 = 0$.
34. $5x - 7y - 3 = 0$.
35. $2xy - 3x - 2y - 4 = 0$.
36. Solve $F = P(1 + rt)$, (*a*) for P; (*b*) for t.

Each equation will reduce to a linear equation if cleared of fractions properly. This reduction may be prevented and extraneous roots may be introduced if unnecessary factors are employed in the L.C.D. *Solve each equation and check each root.*

37. $\dfrac{2x}{x - 1} = 2 + \dfrac{5}{2x}$.

38. $\dfrac{4t - 5}{6t + 1} = \dfrac{2t - 3}{3t + 4}$.

39. $\dfrac{5 + 4t}{1 - 6t} = \dfrac{3}{4 - 3t} + \dfrac{2t}{4 - 3t}$.

40. $\dfrac{1 + 8x}{2x - 1} - \dfrac{20x}{5x - 3} = 0$.

41. $\dfrac{10 - 3x}{21} = \dfrac{4 - 3x}{1 - 9x} - \dfrac{x}{7}$.

42. $\dfrac{t - 3}{t - 1} = \dfrac{t^2 + 20 - 9t}{t^2 + t - 2}$.

43. $\dfrac{2x+1}{3x-1} = \dfrac{2x^2 - x + 14}{3x^2 + 5x - 2}$.

44. $\dfrac{3x-1}{3x-2} = \dfrac{6x^2+6}{6x^2 - x - 2}$.

45. $\dfrac{x+3}{x-1} - \dfrac{2x+3}{x-5} = \dfrac{3-x^2}{x^2-6x+5}$.

46. The Fahrenheit reading, F, and the Centigrade reading, C, in degrees, for a given temperature are related by the equation $F = \frac{9}{5}C + 32$. (*a*) Solve for C in terms of F. (*b*) From the result of Part (*a*), find C if $F = 50°$.

47. Solve $f = ma$ for a.

48. Solve $s = vt$ for t.

49. Solve $s = k + vt$ for v.

50. Solve $t = \pi l/g$ for l.

Solve for x or w, whichever appears.

51. $\dfrac{3x}{2x+n} + \dfrac{2n+x}{2x} = 2$.

52. $\dfrac{w}{b-1} + \dfrac{w}{b+1} = 12$.

53. $\dfrac{x}{.3+x} - .6 = \dfrac{.3-x}{.3+x}$.

54. $\dfrac{aw+b^2}{b^2-a^2} + \dfrac{w+b}{a-b} = \dfrac{w+a}{a+b}$.

Solve for P, or A, correct to two decimal places, by first clearing of fractions.

55. $300 = P[1 + \frac{5}{6}(.07)]$.

56. $500 = A[1 - \frac{7}{12}(.05)]$.

38. Applications of algebra demand that number expressions described in words be translated into formulas.

EXAMPLE 1. How long will it take Jones and Smith, working together, to plow a field which Jones can plow alone in 5 days and Smith, alone, in 8 days?

SOLUTION. 1. Let x days be the time required by Jones and Smith, working together. In 1 day, Jones can plow $\frac{1}{5}$ and Smith $\frac{1}{8}$ of the field. Hence, the fractional part of the field which can be plowed in x days by Jones is $\frac{x}{5}$ and by Smith is $\frac{x}{8}$.

2. Since the *whole* field is plowed in x days, the sum of the fractional parts plowed by the men in x days is 1:

$$\frac{x}{5} + \frac{x}{8} = 1; \quad 8x + 5x = 40; \quad x = 3\tfrac{1}{13} \text{ days}.$$

Note 1. The words *per cent* are abbreviated by the symbol % and mean *hundredths*. To change a number r to per cent form we multiply r by 100 and add the symbol %. Thus, if $r = .0175$, then $r = 1.75\%$. If a number M is described by the relation $M = Nr$, where $r = M/N$, and if r is expressed in per cent form, we then say that M is expressed as a **percentage** of the **base** N.

EXAMPLE 2. What percentage of a 20% solution of hydrochloric acid should be drawn off and replaced by water to give a 15% solution?

SOLUTION. 1. Think of the solution as consisting of 100 units of volume; then the solution contains 20 units of acid.

2. Let $x\%$ be the rate for the percentage which should be drawn off. Then, from the 100 units we should draw off $x\%$ of 100, or x units.

3. In x units there are $.20x$ units of acid. There remain $(20 - .20x)$ units in the final solution of 100 units, after water is added. Therefore,

$$.15 = \frac{20 - .20x}{100}; \qquad 15 = 20 - .20x; \qquad x = 25.$$

Or, we should draw off 25% of the original solution.

Note 2. If a principal $\$P$ is invested for t years at the rate r, **simple interest,** the interest I and the final **amount** F resulting at the end of t years satisfy the equations

$$I = Prt; \qquad F = P + I; \qquad F = P(1 + rt). \tag{1}$$

In (1), if r is in per cent form, we have I expressed as a percentage of the principal, multiplied by t. We may call P the **present value** of the amount F, due at the end of t years. Unless otherwise specified, in this book, the word *interest* will mean *simple interest.*

ILLUSTRATION 1. To find the present value of \$1100 which is due at the end of $2\frac{1}{2}$ years, if money can be invested at 4%, we would solve for P in the equation $F = P(1 + rt)$ with $F = \$1100$, $r = .04$, and $t = 2\frac{1}{2}$.

EXERCISE 9

Solve by use of an equation in just one unknown.

1. A line 68 inches long is divided into two parts where one is 3 inches longer than the other. Find their lengths.

2. Find the dimensions of a rectangle whose perimeter is 55 feet, if the altitude is $\frac{2}{3}$ of the base. (The *perimeter* of a polygon is the sum of the lengths of the sides.)

3. A sum of money amounting to \$13.55 consists of nickels, dimes, and quarters. There are three times as many dimes as nickels and three less quarters than dimes. How many of each coin are there?

4. In 1 hour, Jones can plow $\frac{1}{8}$ of a field and Roberts $\frac{1}{12}$ of it. If they work together, how long will it take them to plow the field?

5. How long will it take two mechanical ditchdiggers to excavate a ditch which the first machine, alone, could complete in 8 days and the second, alone, in 11 days?

6. A merchant has some coffee worth 70¢ per pound and some worth 50¢. How many pounds of each are used in forming 100 pounds of a mixture worth 65¢ per pound?

7. How many gallons of a solution of glycerine and water containing 55% glycerine should be added to 15 gallons of a 20% solution to give a 40% solution?

8. How many pounds of cream containing 35% butterfat should be added to 800 pounds of milk containing 3% butterfat to give milk containing 3.5% butterfat?

9. An airplane and its carrier start at 7 A.M. from the same place, in opposite directions, at speeds of 200 miles and 30 miles per hour, respectively. When will they be 600 miles apart?

10. At 6 A.M., a motorcycle messenger starts from a city at a speed of 45 miles per hour to meet a regiment which is 120 miles away and is approaching at a speed of 5 miles per hour. When will the messenger meet the regiment?

Note 1. The point of support of a lever is called its *fulcrum.* If two or more weights are placed along a lever in such a way that the lever is in a position of equilibrium, then, if each weight is multiplied by its lever arm, the sum of these products for all weights on one side of the fulcrum equals the sum of the products for all weights on the other side. In other words, *the sum of the moments of the weights about the fulcrum is the same on both sides.* In all lever problems in this book, it will be assumed that the weight of the lever is negligible for the purpose in view.

11. A teeterboard is balanced when one girl weighing 80 pounds sits 4 feet from the fulcrum, another girl weighing 100 pounds sits 7 feet from the fulcrum on the other side, and a third girl sits 6 feet from the fulcrum. How much does the third girl weigh?

12. Jones and Smith together weigh 340 pounds. Find their weights if they balance a lever when Jones sits 5 feet from its fulcrum on one side and Smith sits 6 feet from the fulcrum on the other side.

13. How heavy a weight can a man lift with a lever 9 feet long if the fulcrum is 2 feet from the end under the weight and if the man exerts a force of 140 pounds on the other end?

14. At what rate will $750 be the interest for 5 years on $6000?

15. Find the invested principal if it earns $375 interest in 3 months when the interest rate is $3\frac{1}{2}\%$.

16. Find the principal if it earns $150 interest in $\frac{1}{2}$ year at 8%.

17. (*a*) Find the principal which will amount to $1300 by the end of 6 years when invested at 5%. (*b*) Verify the result by computing interest on it for 6 years.

18. Find the present value of $1888 which is due at the end of 4 years, if the interest rate is $4\frac{1}{2}\%$.

19. Jones agreed to pay Smith $6000 at the end of 5 years. What should Jones pay immediately to cancel his debt if Smith agrees that he can invest money at 4%?

20. An airplane leaves the deck of a battleship and travels south at the rate of 230 miles per hour. The battleship travels south at the rate of 20 miles per hour. If the wireless set on the airplane has a range of 500 miles, when will the airplane pass out of wireless communication with the ship?

21. On a river whose current flows at the rate of 3 miles per hour, a motorboat takes as long to travel 12 miles downstream as to travel 8 miles upstream. At what rate could the boat travel in still water? (The rate of the current is added to, or subtracted from, the rate in still water, as the boat goes downstream or upstream, respectively.)

22. Roberts buys a bill of goods from a merchant who asks $2000 at the end of 2 months. If Roberts wishes to pay immediately, what should the seller be willing to accept if he is able to invest his money at 8%?

23. How long will it take a given principal to double itself if invested at 5% simple interest?

24. A man invests $7000, one part of it at 5% and the balance at 4%. If the total annual interest is $320, how much is invested at each rate?

25. When the wind velocity is 40 miles per hour, it takes a certain airplane as long to travel 320 miles against the wind as 480 miles with it. How fast can the airplane fly in still air?

26. An airplane which can fly 250 miles per hour in still air takes $\frac{2}{3}$ as long to make a trip between two cities with the wind as is required for the return journey. Find the velocity of the wind.

27. A tank has one supply pipe which fills it in 6 hours, and another which fills it in 9 hours. How many hours will it take to fill the tank if both pipes are used simultaneously?

28. A boat, which can travel 9 miles per hour in still water, travels upstream for a certain time on a river whose current flows at the rate of 2 miles per hour, and then the boat returns to its starting place. If the trip, up and back, consumed 7 hours, how far did the boat travel upstream?

29. A man makes two investments whose sum is $12,000. In one year he gained 5% interest on one investment and lost 3% on the other; his net gain was $304. Find each of the investments.

30. Find two consecutive positive integers whose squares differ by 27.

31. At how many minutes after 2 P.M. will the minute hand of a clock overtake the hour hand?

32. At how many minutes after 4 P.M. will the hands of a clock become perpendicular for the first time?

CHAPTER TWO

Exponents and Radicals

39. Imaginary numbers. We have called R a *square root* of A if $R^2 = A$. If A is positive, it has exactly *two square roots*, one positive and one negative, denoted by $\pm \sqrt{A}$. If a negative number, $-P$, has R as a square root, then $R^2 = -P$. But, if R is either positive or negative then R^2 is positive and cannot equal $-P$. Hence, $-P$ has *no positive or negative square root.* Therefore, in order that $-P$ may have square roots, we *define* the symbol $\sqrt{-P}$ as a new variety of number, called an **imaginary number,** with the property that

$$(\sqrt{-P})^2 = -P \qquad \text{and} \qquad (-\sqrt{-P})^2 = -P.$$

Thus, $-P$ has the two imaginary numbers $\pm\sqrt{-P}$ as square roots. As an extension of this terminology, we agree that, if M is a real number, each of the expressions $(M + \sqrt{-P})$ and $(M - \sqrt{-P})$ will be called an imaginary number. Unless otherwise stated, any literal number in this book will represent a *real* number.

ILLUSTRATION 1. The square roots of the negative number -5 are the imaginary numbers $\pm\sqrt{-5}$. $(7 + \sqrt{-18})$ is an imaginary number.

40. Roots. We call R a *square root* of A if $R^2 = A$ and a *cube root* of A if $R^3 = A$. If n is any positive integer we say that

$$\textbf{R is an nth root of A if } R^n = A. \tag{1}$$

The following facts will be proved at a more advanced stage.

1. *Every number A, not zero, has just n distinct* nth *roots, some or all of which may be imaginary numbers.*

2. *If n is* **even,** *every positive number A has just* **two** *real* nth *roots, one positive and one negative, with equal absolute values.*

3. *If n is* **odd,** *every real number A has just* **one** *real* nth *root, which is positive when A is positive and negative when A is negative.*

4. *If n is* **even** *and A is* **negative,** *all* nth *roots of A are imaginary numbers.*

ILLUSTRATION 1. The only nth root of 0 is 0. 2 is a 5th root of 32 because $2^5 = 32$. -3 is a cube root of -27.

If A is *positive*, its *positive* nth root is called the **principal nth root** of A. If A is negative and n is odd, the *negative* nth root of A is called its *principal* nth *root.*

ILLUSTRATION 2. The real 4th roots of 81 are ± 3 and $+3$ is the principal 4th root. The principal cube root of $+125$ is $+5$ and of -125 is -5. All 4th roots of -16 are imaginary numbers.

41. Radicals. The *radical* $\sqrt[n]{A}$, which we read *the* nth root of A, is used to denote the *principal* nth root of A when it has a real nth root, and to denote any convenient nth root of A if all nth roots are imaginary. In $\sqrt[n]{A}$, the integer n is called the *index* or *order* of the radical, and A is called its **radicand.** When $n = 2$, we omit the index and use $\sqrt{A}$ instead of $\sqrt[2]{A}$ for the square root of A. In this chapter we shall deal only with *real* nth roots.

I. $\sqrt[n]{A}$ *is positive if A is positive.*

II. $\sqrt[n]{A}$ *is negative if A is negative and n is odd.*

III. $\sqrt[n]{A}$ *is imaginary if A is negative and n is even.*

ILLUSTRATION 1. $\sqrt[4]{81} = 3$. $\sqrt[5]{-32} = -2$. $\sqrt[4]{-8}$ is imaginary.

42. Rational and irrational numbers. A real number which can be expressed as a fraction m/n, where the numerator and denominator are integers, is called a *rational number.* All integers are included among the rational numbers because, if m is any integer, then m can be expressed as the fraction $m/1$. A real number which is *not* a rational number is called an *irrational number.*

ILLUSTRATION 1. 7, 0, and $-\frac{5}{9}$ are rational numbers. Any terminating decimal fraction, such as 3.017, or 3017/1000, is a rational number. π and $\sqrt{2}$ are irrational numbers (see Note 1 in the Appendix). Any irrational number can be expressed as an endless but not a repeating decimal fraction. In particular, $\pi = 3.14159\ldots$ and $\sqrt{2} = 1.414\ldots$ are endless but not repeating decimals. In a later chapter we shall prove that any endless *repeating* decimal represents a *rational* number. Thus, $.1666\ldots = 1/6$.

If A is not the nth power of a rational number, and $\sqrt[n]{A}$ is real, then $\sqrt[n]{A}$ is irrational and is called a **surd** of the nth order.

ILLUSTRATION 2. $\sqrt{3}$ is a surd. $\sqrt{64}$ is not a surd because $\sqrt[6]{64} = 2$. A surd of the second order is sometimes called a *quadratic surd* and one of the third order a *cubic surd.*

43. Rational and irrational expressions. An algebraic expression is said to be *rational* in certain letters if it can be expressed as a fraction whose numerator and denominator are *integral rational polynomials* in the letters. An algebraic expression which is *not* rational in the letters is said to be *irrational* in them. Hereafter, until otherwise stated, in any integral rational polynomial we shall assume that the numerical coefficients are *rational* numbers.

ILLUSTRATION 1. $\dfrac{x^3 - 3a^2 + 1}{2x - 5a}$ is rational in a and x.

ILLUSTRATION 2. $\sqrt{3x + y}$ is not rational in x and y.

ILLUSTRATION 3. The expression $x\sqrt{2} - 3x^2 - \sqrt{5}$ is an integral rational polynomial in x. The presence of $\sqrt{2}$ and $\sqrt{5}$ is of no concern.

44. Perfect powers. A rational expression is called a *perfect nth power* if it is the nth power of some rational expression. A rational number is called a perfect nth power if it is the nth power of some *rational* number.

If an integral rational term is a perfect nth power, then each exponent in the term has n as a factor, because in obtaining the nth power of a monomial each exponent is multiplied by n.

ILLUSTRATION 1. $32y^{15}$ is a perfect 5th power: $\quad 32y^{15} = (2y^3)^5.$

45. Elementary properties of radicals. In the following list, Property I is a direct consequence of the definition of an nth root. The other properties require formal proofs.

I. $\quad (\sqrt[n]{a})^n = a.$

ILLUSTRATION 1. $\quad (\sqrt[5]{17})^5 = 17. \quad (\sqrt[3]{2x^2y})^3 = 2x^2y.$

II. $\quad \sqrt[n]{a^n} = a.$ $\quad$ *(If a is positive when n is even)**

ILLUSTRATION 2. $\quad \sqrt[3]{x^3} = x. \quad \sqrt[4]{5^4} = 5.$

Proof of (II). Since $(a)^n = a^n$, by definition $\sqrt[n]{a^n} = a$.

III. $\quad \sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}.$

ILLUSTRATION 3. $\quad \sqrt{9x^2} = \sqrt{9}\sqrt{x^2} = 3x.$

* To avoid imaginary numbers in elementary problems, the following agreement will hold in this book unless otherwise specified. If the index of a radical is an *even* integer, all literal numbers in the radicand not used as exponents represent *positive numbers*, and are such that the radicand is *positive*.

IV. $$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

ILLUSTRATION 4. $\sqrt[4]{\frac{81}{16}} = \frac{\sqrt[4]{81}}{\sqrt[4]{16}} = \frac{3}{2}$, because $\left(\frac{3}{2}\right)^4 = \frac{3^4}{2^4} = \frac{81}{16}.$

V. *If m/n is an integer,* $\sqrt[n]{a^m} = a^{\frac{m}{n}}.$

ILLUSTRATION 5. $\sqrt[3]{a^{12}} = a^{\frac{12}{3}} = a^4$, because $(a^4)^3 = a^{12}.$

★*Note 1.* The following proofs are complete if a and b are *positive.* The interested student may consider the possibility of negative values for a and b. To complete the proofs, it should be demonstrated that in all cases the two sides of each formula, in (III), (IV), and (V), are either both positive or both negative.

Proof of (III). Raise $(\sqrt[n]{a}\sqrt[n]{b})$ to the nth power:

$$(\sqrt[n]{a}\sqrt[n]{b})^n = (\sqrt[n]{a})^n(\sqrt[n]{b})^n = ab.$$

Hence, by definition, $\sqrt[n]{a}\sqrt[n]{b}$ is an nth root of ab, or $\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}.$

Proof of (V). By Law II, page 6, $(a^{\frac{m}{n}})^n = a^{\frac{m}{n}\cdot n} = a^m$. Hence, $a^{\frac{m}{n}}$ is an nth root of a^m, or $a^{\frac{m}{n}} = \sqrt[n]{a^m}.$

We observe that Property II is a special case of Property V.

ILLUSTRATION 6. $\sqrt[3]{8x^6y^9} = \sqrt[3]{(2x^2y^3)^3} = 2x^2y^3.$ (Property II)

Or, by Properties III and V, $\sqrt[3]{8x^6y^9} = \sqrt[3]{8}\sqrt[3]{x^6}\sqrt[3]{y^9} = 2x^2y^3.$

ILLUSTRATION 7. $\sqrt[4]{\frac{81}{16x^8}} = \frac{\sqrt[4]{81}}{\sqrt[4]{16x^8}} = \frac{3}{2x^2}.$ (Properties IV; II)

Note 2. If a is *negative* and n is even, then a^n is *positive* and the *positive* nth root of a^n is $-a$, or $\sqrt[n]{a^n} = -a$. This case is ruled out in Property II. For instance, if a is negative, $\sqrt{a^2} = -a.$

EXERCISE 10

Perform the operations by use of the index laws.

1. $(x^3)^5.$ 2. $(5x^2y)^4.$ 3. $(-2a^3)^4.$ 4. $(d^2)^{hk}.$ 5. $(-.3cd^3)^2.$

6. $\left(\frac{c}{d}\right)^3.$ 7. $\frac{a^5b^7}{a^3b^9}.$ 8. $\left(\frac{4a^2}{3x}\right)^3.$ 9. $\left(\frac{z^n}{a^h}\right)^3.$ 10. $\left(\frac{c^xd^y}{a^2}\right)^n.$

11. State the two square roots of 64; 49; 81; $\frac{1}{9}$; $\frac{4}{25}$; .01.
12. State the principal square root of 100; 144; $\frac{1}{64}$; $\frac{25}{49}$.
13. State the principal cube root of 8; -27; -1; -216; $\frac{1}{27}$; 125.

14. State the two real fourth roots of 81; 16; 10,000; $\frac{1}{16}$; .0001.

Find the specified power of the radical, or the indicated root.

15. $\sqrt{b^2}$. **16.** $\sqrt[3]{a^3}$. **17.** $\sqrt[4]{3^4}$. **18.** $\sqrt[3]{d^3}$.

19. $(\sqrt{26})^2$. **20.** $(\sqrt[4]{57})^4$. **21.** $(\sqrt[3]{2x})^3$. **22.** $\sqrt{a^4}$.

23. $(\sqrt[5]{-4})^5$. **24.** $(\sqrt[3]{x^2y})^3$. **25.** $\sqrt{36}$. **26.** $\sqrt[3]{x^9}$.

27. $\sqrt[5]{32}$. **28.** $\sqrt[3]{64}$. **29.** $\sqrt[4]{625}$. **30.** $\sqrt[4]{a^{12}}$.

31. $\sqrt[5]{-1}$. **32.** $\sqrt[3]{-27}$. **33.** $\sqrt[3]{8000}$. **34.** $\sqrt[3]{.008}$.

35. $\sqrt[3]{-\frac{1}{8}}$. **36.** $\sqrt[3]{\frac{1}{64}}$. **37.** $\sqrt[3]{\frac{1}{125}}$.

38. $\sqrt[4]{\frac{81}{625}}$. **39.** $\sqrt{\frac{4}{81}}$. **40.** $\sqrt[3]{\frac{8}{27}}$.

41. $\sqrt[5]{y^{10}}$. **42.** $\sqrt[3]{b^6}$. **43.** $\sqrt[3]{8x^3}$.

44. $\sqrt{25x^2}$. **45.** $\sqrt[4]{81a^8}$. **46.** $\sqrt[3]{-8a^3}$.

47. $\sqrt{\frac{x^2}{4}}$. **48.** $\sqrt{\frac{4x^8}{9y^4}}$. **49.** $\sqrt[3]{\frac{a^3b^3}{27}}$. **50.** $\sqrt[3]{-\frac{27}{z^6}}$.

51. $\sqrt{9w^4}$. **52.** $\sqrt{y^4w^6}$. **53.** $\sqrt{9x^8}$. **54.** $\sqrt[3]{x^6y^9}$.

55. $\sqrt{\frac{25}{49}}$. **56.** $\sqrt[3]{27y^9}$. **57.** $\sqrt[4]{16x^4y^8}$. **58.** $\sqrt[5]{-32z^{10}}$.

59. $\sqrt[4]{.0625}$. **60.** $\sqrt[3]{-x^3z^9}$. **61.** $\sqrt[3]{\frac{1}{8}x^3}$. **62.** $\sqrt[3]{216a^6}$.

63. $\sqrt{.04x^{10}}$. **64.** $\sqrt[3]{.125a^3}$. **65.** $\sqrt[3]{-\frac{27}{1000}}$. **66.** $\sqrt[3]{-.064}$.

67. $\sqrt{\frac{25x^4}{4z^{12}}}$. **68.** $\sqrt[3]{\frac{a^6b^6}{27}}$. **69.** $\sqrt[5]{\frac{x^{10}}{y^5}}$. **70.** $\sqrt[3]{\frac{-125}{8z^6}}$. **71.** $\sqrt[3]{\frac{64}{a^3y^6}}$.

72. What is the expression for $\sqrt[4]{x^4}$ when (*a*) x is positive; (*b*) x is negative?

46. Fractional powers. We have previously defined a^p only when p is a positive integer. We shall now introduce other types of powers in such a way that all the types, together, will obey laws of the same forms as those for positive integral exponents.

If fractional exponents are to obey the law of exponents for multiplication, then, for example, we should have

$$a^{\frac{5}{2}}a^{\frac{5}{2}} = a^{\frac{5}{2}+\frac{5}{2}} = a^5, \quad \text{or} \quad (a^{\frac{5}{2}})^2 = a^5,$$

so that $a^{\frac{5}{2}}$ should be a square root of a^5. Accordingly, if m and n are any positive integers, *we define* $a^{\frac{m}{n}}$ **to be the principal nth root of a^m:**

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}; \tag{1}$$

[when $m = 1$ in (1)]

$$a^{\frac{1}{n}} = \sqrt[n]{a}. \tag{2}$$

The defining equation 1 is consistent with Property V of page 35, which was proved for the case where m/n was an *integer*.

ILLUSTRATION 1. $a^{\frac{1}{3}} = \sqrt[3]{a}$. $(-8)^{\frac{2}{3}} = \sqrt[3]{(-8)^2} = \sqrt[3]{64} = 4$.

In accordance with the footnote on page 34, we agree *not to deal with $a^{\frac{m}{n}}$ if n is even and a is negative.* With this case eliminated,

$$\sqrt[n]{a^m} = (\sqrt[n]{a})^m, \tag{3}$$

because $$[(\sqrt[n]{a})^m]^n = [(\sqrt[n]{a})^n]^m = [a]^m = a^m.$$

Hence, from equations 1 and 3, we obtain the new formula

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m. \tag{4}$$

ILLUSTRATION 2. From (4), $64^{\frac{5}{6}} = (\sqrt[6]{64})^5 = 2^5 = 32$. Notice the relative inconvenience of the following evaluation by use of (1):

$$64^{\frac{5}{6}} = \sqrt[6]{64^5} = \sqrt[6]{(2^6)^5} = \sqrt[6]{2^{30}} = 2^5 = 32.$$

★*Note 1.* The following contradiction "$+ 1 = - 1$" results from reckless use of the symbol $(-1)^{\frac{2}{6}}$, which we have agreed not to use.

$$-1 = \sqrt[3]{-1} = (-1)^{\frac{1}{3}} = (-1)^{\frac{2}{6}} = \sqrt[6]{(-1)^2} = \sqrt[6]{+1} = +1.$$

47. Zero as an exponent. If operations with a^0 are to obey the law of exponents for multiplication, then we should define a^0 so that

$$a^0a^n = a^{0+n} = a^n, \quad \text{or} \quad a^0a^n = a^n, \quad \text{or} \quad a^0 = \frac{a^n}{a^n} = 1.$$

Hence, if $a \neq 0$, *we define a^0 by the equation* $\boldsymbol{a^0 = 1}$.

48. Negative exponents. If a negative exponent is to obey the laws of exponents, then, for instance, we should have $a^3a^{-3} = a^{3-3} = a^0 = 1$. Hence, if p is any positive exponent of the types previously introduced, we define a^{-p} by the equation $a^pa^{-p} = 1$, or

$$a^{-p} = \frac{1}{a^p}. \tag{1}$$

ILLUSTRATION 1. $x^{-5} = \dfrac{1}{x^5}$. $5^{-2} = \dfrac{1}{5^2} = \dfrac{1}{25}$. $(-8)^{-\frac{1}{3}} = \dfrac{1}{(-8)^{\frac{1}{3}}} = -\dfrac{1}{2}$.

In a fraction, any power which is a factor of one term (numerator or denominator) may be removed if the factor, with *the sign of its exponent changed,* is written as a factor of the other term. That is,

$$\frac{a}{bx^n} = \frac{ax^{-n}}{b}. \tag{2}$$

Proof. $$\frac{a}{bx^n} = \frac{a \cdot x^{-n}}{bx^n \cdot x^{-n}} = \frac{ax^{-n}}{bx^0} = \frac{ax^{-n}}{b}.$$

49. Extension of the index laws. We have defined a^p if p is any rational number but we have proved the index laws only for positive integral exponents. A detailed discussion (see Appendix, Note 2) shows that the formulas of Laws I to V of pages 6 and 7 apply if the exponents are any rational numbers.

Hereafter, unless otherwise specified, **to simplify** *an expression involving exponents* will mean to perform indicated operations by use of Laws I to V, and to express the result *without radicals and without zero or negative exponents.*

In operations with exponents, it is frequently convenient to express numerical coefficients as products of powers of prime factors.

ILLUSTRATION 1. $(x^6)^{\frac{2}{3}} = x^{6\cdot\frac{2}{3}} = x^4.$ $x^{\frac{1}{4}}x^{\frac{2}{3}} = x^{\frac{1}{4}+\frac{2}{3}} = x^{\frac{11}{12}}.$

$$\left(\frac{216x^5}{125x^{-1}}\right)^{\frac{2}{3}} = \left(\frac{8\cdot 27x^5x}{125}\right)^{\frac{2}{3}} = \left(\frac{2^3 3^3 x^6}{5^3}\right)^{\frac{2}{3}} = \frac{2^2 3^2 x^4}{5^2} = \frac{36x^4}{25}.$$

$$(x^{\frac{2}{3}} - y^{\frac{2}{3}})(x^{\frac{2}{3}} + y^{\frac{2}{3}}) = (x^{\frac{2}{3}})^2 - (y^{\frac{2}{3}})^2 = x^{\frac{4}{3}} - y^{\frac{4}{3}}.$$

ILLUSTRATION 2. $$\frac{a^{-2}y^{-2}}{a^{-2}+y^{-2}} = \frac{\dfrac{1}{a^2y^2}}{\dfrac{1}{a^2}+\dfrac{1}{y^2}} = \frac{\dfrac{1}{a^2y^2}}{\dfrac{y^2+a^2}{a^2y^2}} = \frac{1}{y^2+a^2}.$$

ILLUSTRATION 3. On multiplying both numerator and denominator below by a^2y^2 to eliminate negative exponents, we obtain

$$\frac{a^{-2}y^{-2}}{a^{-2}+y^{-2}} = \frac{(a^{-2}y^{-2})(a^2y^2)}{(a^{-2}+y^{-2})(a^2y^2)} = \frac{a^0y^0}{a^0y^2+a^2y^0} = \frac{1}{y^2+a^2}.$$

EXERCISE 11

Find the value of the symbol by changing to a radical or from a negative to a positive exponent if necessary. Use formula 4, page 37, when convenient.

1. $25^{\frac{1}{2}}$.	**7.** $(\frac{1}{16})^{\frac{1}{2}}$.	**13.** $4^{\frac{3}{2}}$.	**19.** $(171)^0$.	**25.** $4^{\frac{5}{2}}$.
2. $8^{\frac{1}{3}}$.	**8.** $(\frac{1}{27})^{\frac{1}{3}}$.	**14.** $(-27)^{-\frac{1}{3}}$.	**20.** $.0001^{\frac{1}{4}}$.	**26.** $81^{\frac{3}{4}}$.
3. 4^{-1}.	**9.** $(\frac{27}{8})^{\frac{1}{3}}$.	**15.** $(\frac{3}{4})^{-2}$.	**21.** $(-125)^{\frac{1}{3}}$.	**27.** $(\frac{4}{9})^{\frac{3}{2}}$.
4. 16^0.	**10.** $16^{-\frac{1}{2}}$.	**16.** $(-2)^{-3}$.	**22.** $(-\frac{1}{32})^{-\frac{1}{5}}$.	**28.** $(\frac{8}{27})^{\frac{2}{3}}$.
5. 5^{-3}.	**11.** $(-8)^{\frac{1}{3}}$.	**17.** $.36^{\frac{1}{2}}$.	**23.** $8^{\frac{2}{3}}$.	**29.** $(-27)^{\frac{2}{3}}$.
6. $81^{\frac{1}{4}}$.	**12.** $(\frac{2}{5})^{-1}$.	**18.** $.008^{\frac{1}{3}}$.	**24.** $25^{\frac{3}{2}}$.	**30.** $(-64)^{\frac{2}{3}}$

Express with positive exponents.

31. b^{-4}.	**32.** $a^{-3}b^2$.	**33.** $3h^{-4}$.	**34.** $x^{-4}y^{-5}$.	**35.** $4^{-1}ax^{-3}$.
36. $\frac{1}{b^{-4}}$.	**37.** $\frac{a^3}{d^{-2}}$.	**38.** $\frac{a^{-2}b^3}{c^2d^4}$.	**39.** $\frac{3x^{-5}}{2a^{-3}}$.	**40.** $\frac{3^{-2}a^3}{5^{-3}b^{-2}}$.

Write without a denominator by use of negative exponents.

41. $\dfrac{5}{y^4}$. **42.** $\dfrac{5z^2}{y^4}$. **43.** $\dfrac{4a^2}{3xy^3}$. **44.** $\dfrac{b^3}{(1.05)^8}$. **45.** $\dfrac{c}{x - 5y}$.

Express each fractional power as a radical and each radical as a power.

46. $z^{\frac{1}{5}}$. **48.** $5c^{\frac{1}{3}}$. **50.** $\sqrt[7]{b^6}$. **52.** $(6c)^{\frac{1}{5}}$. **54.** $(2x^3)^{\frac{3}{4}}$.

47. $a^{\frac{5}{7}}$. **49.** $ax^{\frac{1}{4}}$. **51.** $\sqrt[5]{a^8}$. **53.** $(2xy)^{\frac{1}{2}}$. **55.** $\sqrt[3]{(a+b)^2}$.

Simplify and, if no letters are involved, evaluate.

56. $x^{\frac{3}{4}}x^{\frac{1}{2}}$. **58.** $(3^4)^{\frac{3}{2}}$. **60.** $(4x^2)^{\frac{3}{2}}$. **62.** $x^{-3}x^5$. **64.** $(x^3)^{-1}$.

57. x^0x^4. **59.** $x^{-2}x^2$. **61.** $(5a^{-2})^3$. **63.** $(x^{\frac{2}{3}})^{\frac{1}{5}}$. **65.** $(b^3)^{-2}$.

66. $(2x^{-2})^3$. **69.** $(3x^{-1}y^3)^2$. **72.** $32^{\frac{3}{5}}$. **75.** $(4^{-3}x^6)^{\frac{2}{3}}$.

67. $(a^2x^3)^{-3}$. **70.** $(6x^{-1}y^{-3})^{-2}$. **73.** $125^{\frac{4}{3}}$. **76.** $(27a^{-3}x^6)^{\frac{4}{3}}$.

68. $(b^{-2}x^2)^{-4}$. **71.** $(5a^{-3}b^2)^{-2}$. **74.** $216^{\frac{2}{3}}$. **77.** $(25x^{-2})^{-\frac{3}{2}}$.

78. $\dfrac{x^5}{x^{\frac{7}{2}}}$. **79.** $\dfrac{a^2}{a^{\frac{10}{3}}}$. **80.** $\dfrac{x^{\frac{1}{3}}}{x^{\frac{5}{6}}}$. **81.** $\dfrac{x^3y^0}{x^{\frac{1}{2}}y^{\frac{2}{3}}}$. **82.** $\dfrac{3x^4y}{5x^{\frac{1}{3}}y^{\frac{5}{3}}}$.

83. $\dfrac{3a^{\frac{1}{3}}y^{\frac{1}{4}}}{6a^{\frac{1}{4}}y^{\frac{1}{2}}}$. **85.** $\dfrac{x^{-3}y^{-5}}{x^{-1}y^2}$. **87.** $\left(\dfrac{2a^3x}{3a^{\frac{1}{2}}}\right)^3$. **89.** $\left(\dfrac{3ax^{-1}}{a^{\frac{1}{2}}x^{-2}}\right)^2$.

84. $\dfrac{a^{-1}b^{-2}}{a^{-2}b^{-1}}$. **86.** $\left(\dfrac{2x^2}{5y^{\frac{1}{3}}}\right)^2$. **88.** $\left(\dfrac{x^{\frac{1}{2}}y^{\frac{2}{3}}}{2a^{\frac{1}{2}}}\right)^3$. **90.** $\left(\dfrac{a^4b^{-4}}{16x^{\frac{4}{3}}}\right)^{\frac{1}{4}}$.

91. $(27m^6)^{\frac{2}{3}}$. **92.** $(32a^5b^5)^{\frac{3}{5}}$. **93.** $(x^{3k}y^{5k})^{\frac{1}{k}}$. **94.** $(216x^{-6})^{\frac{2}{3}}$.

Simplify to a single fraction in lowest terms.

95. $a^{-1} + b^{-1}$. **96.** $3a^{-2} + b$. **97.** $a^{-3} - b^{-3}$. **98.** $(4a^{-3} - b)^{-1}$.

99. $\dfrac{a^{-2} - b}{a^{-2} + b}$. **100.** $\dfrac{c^{-2}y^{-2}}{c^{-2} + y^{-2}}$. **101.** $\dfrac{4^{-1} - a^{-1}}{4^{-2} - a^{-2}}$. **102.** $\dfrac{a^{-1} - b^{-1}}{a^{-3} - b^{-3}}$.

Expand and simplify, without eliminating negative exponents.

103. $(x^{-1} - y^{-1})(x^{-1} + y^{-1})$. **105.** $(x^{\frac{1}{4}} + y^{\frac{1}{4}})(x^{\frac{1}{4}} - y^{\frac{1}{4}})$.

104. $(x^{\frac{2}{3}} - 4y)(x^{\frac{2}{3}} + 4y)$. **106.** $(a^{\frac{1}{3}} - 2b^{\frac{1}{3}})(a^{\frac{1}{3}} + 3b^{\frac{1}{3}})$.

107. $(a^{\frac{1}{3}} + b)^2$. **108.** $(x^{\frac{1}{2}} + 3)^2$. **109.** $(x^4 - 2y^{-1})^2$. **110.** $(2 + x^{-2})^3$.

111. $(a - b^{\frac{1}{2}})(a^2 + ab^{\frac{1}{2}} + b)$. **112.** $(c^{\frac{1}{3}} + d)(c^{\frac{2}{3}} - c^{\frac{1}{3}}d + d^2)$.

★*Factor into two factors involving fractional or negative exponents. When possible, factor as the difference of two squares or as a perfect square.*

ILLUSTRATION 1. $x - y = (x^{\frac{1}{2}})^2 - (y^{\frac{1}{2}})^2 = (x^{\frac{1}{2}} - y^{\frac{1}{2}})(x^{\frac{1}{2}} + y^{\frac{1}{2}})$.

113. $x^2 - y^{-2}$. **114.** $y^4 - x^{\frac{2}{3}}$. **115.** $4a - 9b$. **116.** $8a - 27b$.

117. $x^2 - 2xy^{-1} + y^{-2}$.

118. $z^2 - 6zx^{-1} + 9x^{-2}$.

119. $4a^{\frac{2}{3}} - 20a^{\frac{1}{3}}b^{\frac{1}{3}} + 25b^{\frac{2}{3}}$.

120. $3x^{-2} + x^{-1}y - 2y^2$.

★*Find the quotient by use of factoring.*

121. $\dfrac{x^{-4} - y^{-4}}{x^{-2} + y^{-2}}$. **122.** $\dfrac{4x - y}{2x^{\frac{1}{2}} + y^{\frac{1}{2}}}$. **123.** $\dfrac{a - b}{a^{\frac{1}{3}} - b^{\frac{1}{3}}}$. **124.** $\dfrac{a^{3h} - b^{3h}}{a^h - b^h}$.

50. Operations on radicals. Although it is possible to express a radical as a power with a fractional exponent, in some operations it is convenient to retain the radical form.

To remove factors from the radicand in a radical of order n, first separate the radicand into factors of which as many as possible are *perfect* nth *powers*. Then, find the nth root of each of these powers and use the property $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$.

ILLUSTRATION 1. $\sqrt{147} = \sqrt{49 \cdot 3} = \sqrt{49}\,\sqrt{3} = 7\sqrt{3}$.

$$\sqrt[5]{64a^{10}c^9} = \sqrt[5]{2^6a^{10}c^9} = \sqrt[5]{2^5a^{10}c^5 \cdot 2c^4} = \sqrt[5]{2^5a^{10}c^5}\,\sqrt[5]{2c^4} = 2a^2c\sqrt[5]{2c^4}.$$

In a sum, two or more terms involving the same radical as a factor may be combined by factoring.

ILLUSTRATION 2. $5\sqrt{3} + 2b\sqrt{3} = (5 + 2b)\sqrt{3}$.

The product or the quotient of two radicals of the same order can be expressed as a single radical:

$$\sqrt[n]{a}\,\sqrt[n]{b} = \sqrt[n]{ab}; \qquad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}.$$

ILLUSTRATION 3. $(2\sqrt{3})(5\sqrt{6}) = 10\sqrt{3}\,\sqrt{6} = 10\sqrt{18} = 30\sqrt{2}$.

ILLUSTRATION 4. $\dfrac{\sqrt{3}}{\sqrt{5}} = \sqrt{\dfrac{3}{5}}$. $\qquad \dfrac{\sqrt[3]{ab}}{\sqrt[3]{b^5}} = \sqrt[3]{\dfrac{a}{b^4}} = \dfrac{1}{b}\sqrt[3]{\dfrac{a}{b}}$.

ILLUSTRATION 5. $(2\sqrt{3} + 3\sqrt{2x})(3\sqrt{3} - \sqrt{2x})$

$$= 6(\sqrt{3})^2 - 2\sqrt{3}\,\sqrt{2x} + 9\sqrt{3}\,\sqrt{2x} - 3(\sqrt{2x})^2$$
$$= 18 - 2\sqrt{6x} + 9\sqrt{6x} - 3(2x) = 18 - 6x + 7\sqrt{6x}.$$

Comment. The student may prefer an expanded solution:

$$\begin{array}{l}
\quad 2\sqrt{3} + 3\sqrt{2x} \\
\quad \underline{3\sqrt{3} - \sqrt{2x}} \qquad (\textit{multiply}) \\
6(\sqrt{3})^2 + 9\sqrt{3}\,\sqrt{2x} \\
\underline{\qquad\quad - 2\sqrt{3}\sqrt{2x} - 3(\sqrt{2x})^2} \\
\ 18 \quad + 7\sqrt{3}\,\sqrt{2x} - 3(2x) = 18 - 6x + 7\sqrt{6x}.
\end{array}$$

ILLUSTRATION 6. $\sqrt[3]{2a}\sqrt[3]{20a^2b^3} = \sqrt[3]{40a^3b^3} = \sqrt[3]{8a^3b^3}\sqrt[3]{5} = 2ab\sqrt[3]{5}.$

If we remove a positive factor P multiplying a radical $\sqrt[n]{A}$ we must *multiply by P^n under the radical sign* because

$$P\sqrt[n]{A} = \sqrt[n]{P^n}\sqrt[n]{A} = \sqrt[n]{P^nA}.$$

ILLUSTRATION 7. $3\sqrt{b} = \sqrt{9}\sqrt{b} = \sqrt{9b}.$

Hereafter, unless otherwise specified, if a radicand involves fractions, reduce it to a *single* fraction. If a radical is of order n, simplify the radicand by removing from it any factor which is a perfect nth power. Also, in a radical of *odd* order, change the radicand to a form where all signs are "+" if possible.

ILLUSTRATION 8. $\sqrt[3]{3a + \dfrac{5}{x^3}} = \sqrt[3]{\dfrac{3ax^3 + 5}{x^3}} = \dfrac{\sqrt[3]{3ax^3 + 5}}{\sqrt[3]{x^3}} = \dfrac{\sqrt[3]{3ax^3 + 5}}{x}.$

ILLUSTRATION 9. $\sqrt[3]{-2} = \sqrt[3]{-1}\sqrt[3]{2} = -1 \cdot \sqrt[3]{2} = -\sqrt[3]{2}.$

$$\sqrt[3]{-a-2b} = \sqrt[3]{-(a+2b)} = \sqrt[3]{-1}\sqrt[3]{a+2b} = -\sqrt[3]{a+2b}.$$

EXERCISE 12

Simplify by removing perfect powers from the radicands. If no letters are involved, then compute by use of Table I.

1. $\sqrt{20}.$
2. $\sqrt{24}.$
3. $\sqrt{27}.$
4. $\sqrt{72}.$
5. $\sqrt{.45}.$
6. $\sqrt[3]{24}.$
7. $\sqrt[3]{108}.$
8. $\sqrt[3]{-3}.$
9. $\sqrt[3]{-54}.$
10. $\sqrt[3]{.024}.$
11. $\sqrt{a^9}.$
12. $\sqrt[3]{y^5}.$
13. $\sqrt[4]{x^{10}}.$
14. $\sqrt{9a^4}.$
15. $\sqrt[3]{8a^4}.$
16. $\sqrt[4]{16z^7}.$
17. $\sqrt{12a^3y^5}.$
18. $\sqrt[3]{12a^3y^5}.$
19. $\sqrt[3]{54y^5}.$
20. $\sqrt[3]{16y^8z^4}.$
21. $\sqrt[3]{-27a^8}.$
22. $\sqrt[3]{-a^5y^6}.$
23. $\sqrt[4]{3x^4y^{10}}.$
24. $\sqrt[4]{16c^3d^5}.$
25. $\sqrt[3]{32a^4y^{16}}.$
26. $\sqrt{.04a^9}.$
27. $\sqrt{.25x^7}.$
28. $\sqrt{\dfrac{4x^5}{y^4}}.$
29. $\sqrt[3]{\dfrac{5x^5}{8y^6}}.$
30. $\sqrt[3]{\dfrac{216x^5}{125y^3}}.$
31. $\sqrt[3]{-\dfrac{16}{a^3b^6}}.$
32. $\sqrt{a^2 + 5a^2b}.$
33. $\sqrt[3]{27x^3 - x^3z^3}.$
34. $\sqrt[4]{5x^{4k}}.$
35. $\sqrt[4]{16a^{8n}}.$
36. $\sqrt{\dfrac{3a}{x} + \dfrac{5b}{x^2}}.$
37. $\sqrt{\dfrac{c}{xy^2} + \dfrac{2d}{x^2}}.$
38. $\sqrt[3]{\dfrac{a}{8} - \dfrac{3}{y^3}}.$

Simplify and collect terms, exhibiting any common radical factor.

39. $5\sqrt{2} + 3\sqrt{2}.$
40. $3\sqrt{3} - a\sqrt{3}.$
41. $\sqrt[3]{24} + 2\sqrt[3]{81}.$
42. $\sqrt{9a} + \sqrt{25a}.$
43. $\sqrt[3]{81x} - \sqrt[3]{3x^4}.$
44. $\sqrt[3]{8x^3y} - \sqrt[3]{27a^3y}.$

Express by means of a single radical, simplify, and then compute by use of Table I *if no letters are involved.*

45. $\sqrt{2}\sqrt{3}$. **47.** $\sqrt{3}\sqrt{15}$. **49.** $(3\sqrt{5})^2$. **51.** $\sqrt[3]{-4}\sqrt[3]{18}$.

46. $\sqrt[3]{5}\sqrt[3]{50}$. **48.** $\sqrt{30}\sqrt{35}$. **50.** $(3\sqrt[3]{2})^3$. **52.** $\sqrt[3]{-2}\sqrt[3]{12}$.

53. $\dfrac{\sqrt{15}}{\sqrt{5}}$. **55.** $\dfrac{\sqrt[3]{99}}{\sqrt[3]{11}}$. **57.** $\dfrac{\sqrt{10x}}{\sqrt{2x}}$. **59.** $\dfrac{\sqrt[3]{5h^4}}{\sqrt[3]{40h}}$.

54. $\dfrac{\sqrt[3]{10}}{\sqrt[3]{5}}$. **56.** $\dfrac{\sqrt[3]{-81}}{\sqrt[3]{3}}$. **58.** $\dfrac{\sqrt{3x^3}}{\sqrt{12x}}$. **60.** $\dfrac{\sqrt[3]{6a^5}}{\sqrt[3]{2a^8}}$.

61. $\sqrt{3}\sqrt{x}\sqrt{6x^2}$. **62.** $\sqrt[3]{9a^2}\sqrt[3]{6a^3b^4}$. **63.** $(2\sqrt[3]{4a})^3$. **64.** $(3\sqrt[3]{2x})^3$.

Perform the operation and collect similar terms.

65. $(2+3\sqrt{5})(3-\sqrt{5})$. **67.** $(3\sqrt{2}+\sqrt{3})(\sqrt{2}+4\sqrt{3})$.

66. $(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})$. **68.** $(2\sqrt{3}-\sqrt{7})(2\sqrt{3}+\sqrt{7})$.

69. $(\sqrt{2}+5)^2$. **71.** $(\sqrt{2}-2\sqrt{3})^2$. **73.** $\sqrt[5]{x^2y^4z^5}\sqrt[5]{-x^3yz^4}$.

70. $(3+5\sqrt{2})^2$. **72.** $(\sqrt{x}-5\sqrt{y})^2$. **74.** $\sqrt[3]{-3a^2b}\sqrt[3]{18a^4b^5}$.

Replace the coefficient of the radical by a factor under the radical sign.

75. $3\sqrt{2a}$. **76.** $a\sqrt{bx}$. **77.** $3\sqrt[3]{b^2}$. **78.** $2\sqrt[4]{3b}$.

51. Rationalizing denominators. In a radical of order n, suppose that the radicand has been expressed as a simple fraction. Then, the denominator can be rationalized by multiplying both numerator and denominator of the radicand by the simplest expression which will make the denominator a perfect nth power.

Illustration 1. $\sqrt{\dfrac{3}{7}}=\sqrt{\dfrac{3\cdot 7}{7^2}}=\dfrac{\sqrt{21}}{7}=\dfrac{4.583}{7}=.655.$ (Table I)

Illustration 2. $\sqrt[3]{\dfrac{64}{9x^4}}=\sqrt[3]{\dfrac{4^3\cdot 3x^2}{9x^4\cdot 3x^2}}=\dfrac{\sqrt[3]{4^3\cdot 3x^2}}{\sqrt[3]{3^3x^6}}=\dfrac{4\sqrt[3]{3x^2}}{3x^2}.$

$$\sqrt{a^{-2}+b^{-3}}=\sqrt{\frac{1}{a^2}+\frac{1}{b^3}}=\sqrt{\frac{b^3+a^2}{a^2b^3}}=\sqrt{\frac{b(b^3+a^2)}{a^2b^4}}=\frac{\sqrt{b(b^3+a^2)}}{ab^2}.$$

The method of the following illustration frequently is equivalent to the procedure of Illustration 1.

Illustration 3. The denominator below is multiplied by $\sqrt[3]{2}$ in order to make the new radicand in the denominator a perfect cube:

$$\frac{\sqrt[3]{7}}{\sqrt[3]{4}}=\frac{\sqrt[3]{7}}{\sqrt[3]{4}}\cdot\frac{\sqrt[3]{2}}{\sqrt[3]{2}}=\frac{\sqrt[3]{14}}{\sqrt[3]{8}}=\frac{\sqrt[3]{14}}{2}.$$

ILLUSTRATION 4. $\dfrac{\sqrt{3}}{\sqrt{5}} = \dfrac{\sqrt{3}}{\sqrt{5}} \cdot \dfrac{\sqrt{5}}{\sqrt{5}} = \dfrac{\sqrt{15}}{5}.$

If a denominator has the form $a\sqrt{b} - c\sqrt{d}$, we can rationalize it by multiplying by $a\sqrt{b} + c\sqrt{d}$ because

$$(a\sqrt{b} - c\sqrt{d})(a\sqrt{b} + c\sqrt{d}) = (a\sqrt{b})^2 - (c\sqrt{d})^2 = a^2b - c^2d.$$

ILLUSTRATION 5. $\dfrac{3\sqrt{2} - \sqrt{3}}{2\sqrt{2} - \sqrt{3}} = \dfrac{3\sqrt{2} - \sqrt{3}}{2\sqrt{2} - \sqrt{3}} \cdot \dfrac{2\sqrt{2} + \sqrt{3}}{2\sqrt{2} + \sqrt{3}}$

$$= \frac{6(\sqrt{2})^2 + (3 - 2)\sqrt{6} - (\sqrt{3})^2}{(2\sqrt{2})^2 - (\sqrt{3})^2} = \frac{9 + \sqrt{6}}{8 - 3} = \frac{9 + 2.449}{5} = 2.290.$$

In finding the quotient of two radicals, it may be desirable to write the expression as a single radical before rationalizing.

ILLUSTRATION 6. $\dfrac{\sqrt[3]{3x^3y^7}}{\sqrt[3]{81x^5y^5}} = \sqrt[3]{\dfrac{3x^3y^7}{81x^5y^5}} = \sqrt[3]{\dfrac{y^2}{27x^2}} = \dfrac{\sqrt[3]{xy^2}}{3x}.$

EXERCISE 13

Eliminate any negative exponents and rationalize any denominator. If no letters are involved, then compute by use of Table I.

1. $\sqrt{\frac{1}{3}}$. 2. $\sqrt{\frac{3}{5}}$. 3. $\sqrt{\frac{2}{27}}$. 4. $\sqrt[3]{\frac{1}{4}}$. 5. $\sqrt[3]{-\frac{7}{100}}$. 6. $\sqrt[3]{\frac{5}{16}}$. 7. $\sqrt[3]{\frac{1}{10}}$. 8. $\sqrt[3]{-.03}$. 9. $\sqrt{.012}$. 10. $\sqrt[3]{-.128}$.

11. $\dfrac{1}{\sqrt{3}}$. 12. $\dfrac{1}{\sqrt{5}}$. 13. $\dfrac{6}{\sqrt{5}}$. 14. $\dfrac{3}{\sqrt{7}}$. 15. $\dfrac{2}{\sqrt{3}}$. 16. $\dfrac{3}{\sqrt{11}}$. 17. $\dfrac{\sqrt{5}}{\sqrt{3}}$. 18. $\dfrac{\sqrt{7}}{\sqrt{2}}$. 19. $\dfrac{2\sqrt{3}}{\sqrt{5}}$. 20. $\dfrac{3\sqrt{7}}{2\sqrt{5}}$. 21. $\dfrac{3}{\sqrt[3]{4}}$. 22. $\dfrac{1}{\sqrt[3]{100}}$.

23. $\dfrac{2 - \sqrt{3}}{3 + \sqrt{3}}$. 24. $\dfrac{1}{3 + 2\sqrt{2}}$. 25. $\dfrac{\sqrt{2} - 3}{2\sqrt{2} + \sqrt{3}}$. 26. $\dfrac{\sqrt{3} - \sqrt{5}}{2\sqrt{3} - \sqrt{5}}$. 27. $\dfrac{\sqrt{2} + \sqrt{5}}{3\sqrt{5} - \sqrt{2}}$. 28. $\dfrac{\sqrt{7} - \sqrt{6}}{2\sqrt{6} + \sqrt{7}}$.

29. $\sqrt{\dfrac{x}{3}}$. 30. $\sqrt{\dfrac{2}{a}}$. 31. $\sqrt{\dfrac{1}{3x^2}}$. 32. $\sqrt[3]{\dfrac{c}{4}}$. 33. $\sqrt[3]{\dfrac{a}{2b}}$. 34. $\sqrt{\dfrac{3x^3}{8y^2}}$. 35. $\sqrt[3]{\dfrac{b}{9a^4}}$. 36. $\sqrt[3]{\dfrac{d}{8c^2}}$. 37. $\sqrt[4]{\dfrac{a}{8}}$. 38. $\sqrt[4]{\dfrac{cx^2}{27}}$.

39. $\dfrac{\sqrt[3]{2}}{\sqrt[3]{49}}$. 40. $\dfrac{\sqrt{20x}}{\sqrt{60x^3}}$. 41. $\dfrac{\sqrt{\frac{1}{2}ab^2}}{\sqrt{5a^2b^3}}$. 42. $\dfrac{\sqrt[3]{\frac{1}{4}bc}}{\sqrt[3]{\frac{1}{2}b^2c^2}}$.

43. $\sqrt[3]{\dfrac{-1}{5a^3b^3}}$. 44. $\sqrt[3]{\dfrac{-4ab}{9xz^7}}$. 45. $\sqrt{\dfrac{4h}{a+x}}$. 46. $\sqrt{\dfrac{a-b}{a+b}}$.

47. $\sqrt{x^{-5}}$. 48. $\sqrt{b^{-3}}$. 49. $\sqrt[3]{x^{-2}}$. 50. $\sqrt[3]{a^{-7}b^{-2}}$. 51. $\sqrt{\frac{1}{5}x^{-3}}$.

52. $\sqrt{\dfrac{a}{3}+\dfrac{5}{x}}$. 54. $\sqrt{16+\dfrac{3}{7x^3}}$. 56. $\dfrac{\sqrt{3a}-1}{\sqrt{a+b}-\sqrt{2a}}$.

53. $\sqrt{\dfrac{1}{b}-\dfrac{9}{5a^2}}$. 55. $\dfrac{\sqrt{a}}{\sqrt{a}-\sqrt{c}}$. 57. $\dfrac{3}{\sqrt{3}-\sqrt{2}+\sqrt{5}}$.

58. $\sqrt[n]{\dfrac{3}{x^2}}$. 59. $\sqrt[k]{\dfrac{a}{2b^3}}$. 60. $\sqrt[h]{\dfrac{b^hx}{a^3x^{2h}}}$. 61. $\sqrt[m]{\dfrac{2x^{3m}}{5a^2y^3}}$.

52. To change a product of powers involving fractional exponents to radical form, first change the fractional parts of the exponents to fractions with their lowest common denominator.

ILLUSTRATION 1. $5y^{\frac{2}{3}}z^{\frac{5}{2}} = 5z^2y^{\frac{2}{3}}z^{\frac{1}{2}} = 5z^2y^{\frac{4}{6}}z^{\frac{3}{6}} = 5z^2(y^4z^3)^{\frac{1}{6}} = 5z^2\sqrt[6]{y^4z^3}$.

53. Operations on radicals performed by using fractional exponents, with the results desired in *radical* form.

TYPE I. *To find a power or a root of a radical.*

ILLUSTRATION 1. $\sqrt[3]{\sqrt[4]{3xy}} = [(3xy)^{\frac{1}{4}}]^{\frac{1}{3}} = (3xy)^{\frac{1}{12}} = \sqrt[12]{3xy}$.

In simple cases, it may be unnecessary to introduce fractional exponents in an operation of Type I. Also, with experience, one observes simple rules such as

"*the* mth *root of the* nth *root of A is the* mnth *root of A.*" (1)

ILLUSTRATION 2. $(\sqrt[3]{3})^3 = 3.$ $\sqrt[3]{\sqrt{a}} = \sqrt[6]{a}.$ $(\sqrt[5]{2})^2 = \sqrt[5]{2}\sqrt[5]{2} = \sqrt[5]{4}.$

$$(\sqrt{3x})^5 = [(3x)^{\frac{1}{2}}]^5 = (3x)^{\frac{5}{2}} = (3x)^2(3x)^{\frac{1}{2}} = 9x^2\sqrt{3x}.$$

TYPE II. *To find the product or the quotient of two radicals of different orders, express each radical as a fractional power of its radicand, change the fractions to their* L.C.D., *and then rewrite in radical form.*

ILLUSTRATION 3. $\sqrt{3}\sqrt[5]{2} = 3^{\frac{1}{2}}2^{\frac{1}{5}} = 3^{\frac{5}{10}}2^{\frac{2}{10}} = \sqrt[10]{3^5}\sqrt[10]{2^2} = \sqrt[10]{3^52^2}.$

ILLUSTRATION 4. $\dfrac{\sqrt[3]{4b^2x}}{\sqrt{3ab}} = \dfrac{(4b^2x)^{\frac{1}{3}}}{(3ab)^{\frac{1}{2}}} = \dfrac{(4b^2x)^{\frac{2}{6}}}{(3ab)^{\frac{3}{6}}} = \sqrt[6]{\dfrac{(4b^2x)^2}{(3ab)^3}}$

$$= \sqrt[6]{\frac{16bx^2}{27a^3}} = \frac{\sqrt[6]{27\cdot 16a^3bx^2}}{3a}.$$

TYPE III. *To reduce the order of a radical, when possible, first change to fractional exponents in lowest possible terms with a common denominator, and then rewrite in radical form.*

ILLUSTRATION 5. $\sqrt[6]{625} = \sqrt[6]{5^4} = 5^{\frac{4}{6}} = 5^{\frac{2}{3}} = \sqrt[3]{5^2} = \sqrt[3]{25}.$

$$\sqrt[12]{x^2y^8} = (x^2y^8)^{\frac{1}{12}} = x^{\frac{2}{12}}y^{\frac{8}{12}} = x^{\frac{1}{6}}y^{\frac{4}{6}} = \sqrt[6]{xy^4}.$$

In reducing the order of a radical, it is convenient to commence by expressing the radicand as a power of some expression.

ILLUSTRATION 6. $\sqrt[8]{16x^2} = \sqrt[8]{(4x)^2} = (4x)^{\frac{2}{8}} = (4x)^{\frac{1}{4}} = \sqrt[4]{4x}.$

Note 1. If we are permitted to express the final result of an operation of Type I or of Type II by use of fractional exponents, the operation is fundamentally simple. Fortunately, this exponential form of the result is adequate in a great many applications.

54. Simplest radical form. As far as problems in this text are concerned, we agree that an expression is in its *simplest radical form* if all possible operations of the following varieties have been performed, with any negative exponents eliminated.

1. *Express any power or root of a radical, or product of radicals, as a single radical, with the radicand a simple fraction in lowest terms.*
2. *Rationalize all denominators.*
3. *Remove from each radicand all factors which are perfect* n*th powers, where* n *is the order of the radical.*
4. *Reduce each radical to the lowest possible order.*

The preceding operations need not be performed in the specified order. *To simplify* a radical expression will mean to reduce it to *simplest radical form.*

ILLUSTRATION 1. To simplify the following radical we rationalize the denominator, and finally notice that the order of the radical can be reduced.

$$\sqrt[6]{\frac{a^2}{16c^{10}}} = \sqrt[6]{\frac{a^2 \cdot 4c^2}{16 \cdot 4c^{12}}} = \frac{\sqrt[6]{4a^2c^2}}{2c^2} = \frac{\sqrt[6]{(2ac)^2}}{2c^2} = \frac{\sqrt[3]{2ac}}{2c^2}.$$

EXERCISE 14

Change to simplest radical form.

1. $a^{\frac{1}{3}}b^{\frac{1}{2}}$.	**3.** $5a^{\frac{3}{4}}$.	**5.** $ab^{\frac{2}{3}}$.	**7.** $2a^{\frac{2}{3}}b^{\frac{1}{2}}$.	**9.** $ax^{\frac{1}{2}}y^{\frac{3}{4}}$.	**11.** $2a^2b^{\frac{5}{2}}c^{\frac{5}{6}}$.
2. $x^{\frac{2}{3}}y^{\frac{1}{2}}$.	**4.** $2x^{\frac{3}{7}}$.	**6.** $cd^{\frac{3}{5}}$.	**8.** $3x^{\frac{1}{5}}y^{\frac{2}{3}}$.	**10.** $cd^{\frac{1}{3}}y^{\frac{1}{4}}$.	**12.** $3x^2y^{\frac{4}{3}}z^{\frac{3}{4}}$.

Reduce to a radical of lower order.

13. $\sqrt[4]{x^2}$.
14. $\sqrt[6]{y^2}$.
15. $\sqrt[12]{y^8}$.
16. $\sqrt[6]{z^3}$.
17. $\sqrt[9]{z^3}$.
18. $\sqrt[10]{x^4}$.
19. $\sqrt[4]{9}$.
20. $\sqrt[6]{27}$.
21. $\sqrt[8]{36}$.
22. $\sqrt[8]{49}$.
23. $\sqrt[10]{32}$.
24. $\sqrt[6]{81}$.
25. $\sqrt[4]{9a^2}$.
26. $\sqrt[6]{8a^3}$.
27. $\sqrt[6]{27b^3}$.

Change to simplest radical form.

28. $(\sqrt{b})^3$.
29. $(\sqrt{c})^4$.
30. $(\sqrt[3]{z})^4$.
31. $(\sqrt[3]{x})^2$.
32. $(\sqrt{3})^4$.
33. $(\sqrt{2})^3$.
34. $(\sqrt[3]{2})^5$.
35. $(\sqrt{5})^5$.
36. $(\sqrt{3a})^4$.
37. $(\sqrt{2x})^5$.
38. $(\sqrt[3]{2x})^4$.
39. $(2\sqrt[3]{3})^4$.
40. $\sqrt[3]{\sqrt{x}}$.
41. $\sqrt[5]{\sqrt{z}}$.
42. $\sqrt[3]{\sqrt[5]{5}}$.
43. $\sqrt{\sqrt{3}}$.
44. $\sqrt[3]{\sqrt[3]{a}}$.
45. $\sqrt[4]{\sqrt[4]{x}}$.
46. $\sqrt{\sqrt[3]{x^2}}$.
47. $\sqrt[3]{\sqrt[4]{a^3}}$.
48. $\sqrt{a}\sqrt[3]{a}$.
49. $\sqrt[3]{y}\sqrt[4]{y}$.
50. $\sqrt{3}\sqrt[4]{3}$.
51. $\sqrt[4]{a^2}\sqrt[3]{a^2}$.
52. $\sqrt[3]{a} \div \sqrt{a}$.
53. $\sqrt[4]{b} \div \sqrt{b}$.
54. $\sqrt{2} \div \sqrt[3]{2}$.
55. $\sqrt[3]{5} \div \sqrt[4]{5}$.
56. $\sqrt[6]{8} \div \sqrt{2}$.
57. $\sqrt[4]{3} \div \sqrt[8]{81}$.
58. $\sqrt{6} \div \sqrt[3]{16}$.
59. $\sqrt[4]{cd^3} \div \sqrt{cd}$.
60. $3\sqrt{y} \div \sqrt[3]{x^2y^2}$.
61. $\sqrt[4]{\frac{4}{25}}$.
62. $\sqrt[6]{\frac{27}{8}}$.
63. $\sqrt[3]{-\frac{1}{4}x^5}$.
64. $\sqrt[4]{\frac{1}{9}x^{-3}}$.
65. $\sqrt{x^{-5}}$.
66. $\sqrt[3]{216}$.
67. $\sqrt[3]{\sqrt[4]{8a^3}}$.
68. $\sqrt[3]{27\sqrt{a^3}}$.
69. $\sqrt{9\sqrt[3]{4x^2}}$.
70. $\sqrt[3]{x^2\sqrt{8}}$.
71. $\sqrt{3}\sqrt[3]{3}\sqrt[4]{3}$.
72. $\sqrt[3]{x^{-2}+y^{-2}}$.
73. $\sqrt[6]{\dfrac{a^8}{b^2}}$.
74. $\sqrt[6]{\dfrac{a^{10}}{4b^2}}$.
75. $\sqrt[4]{\dfrac{9x^2y^6}{4b^8}}$.
76. $\sqrt[6]{\dfrac{b^{10}a^4}{9x^2}}$.
77. $(\sqrt[3]{ax^2})^7$.
78. $(2\sqrt[3]{5})^4$.
79. $\sqrt{\sqrt{\sqrt{2}}}$.
80. $\sqrt{\sqrt[3]{9y^2}}$.
81. $(a\sqrt[3]{2})^5$.
82. $(b\sqrt[4]{3})^6$.
83. $\sqrt[6]{2a^4}\sqrt[3]{2a}$.
84. $\sqrt[6]{32}\sqrt{2}$.
85. $\sqrt[3]{-x} - \sqrt[3]{-\dfrac{b^6}{8x^2}}$.
86. $b\sqrt{\dfrac{a}{b}} + a\sqrt{\dfrac{a}{b}+\dfrac{b}{a}+2}$.
87. $\dfrac{2}{\sqrt[4]{9}-\sqrt[4]{4}}$.
88. $\dfrac{3}{\sqrt[4]{9a^2}-\sqrt{b}}$.
89. $\dfrac{\sqrt{3a}+1}{\sqrt{a+b}-\sqrt{3a}}$.
90. $\sqrt[4]{x^{-2}y^{-2}-6x^{-1}y^{-2}+9y^{-2}}$.
91. $\sqrt[6]{64a^2x^6} - \sqrt[9]{a^3b^9}$.
92. $\sqrt{a+b} \div \sqrt[3]{a^2-b^2}$.
93. $\sqrt[8]{81a^8x^4} - \sqrt[4]{9x^2z^4}$.
94. $\sqrt{(a+b)^3} + 2a\sqrt[4]{a^2b^4+2ab^5+b^6} + \sqrt{a^{-1}+b^{-1}}$.
95. $\sqrt{\dfrac{a+1}{a-1}} + \dfrac{1}{1-a^2}\sqrt{4-\dfrac{4}{a^2}}$.
96. $\dfrac{\sqrt[4]{4a^3}}{\sqrt[4]{a-2ab^2+ab^4}}$.

CHAPTER THREE

Rectangular Coordinates and Graphs

55. A number scale. In Figure 1, all positive numbers are represented in order by the points on the scale to the right of A and all negative numbers by the points to the left of A. The unit points on the scale represent the integers and the other points represent the numbers which are not integers.

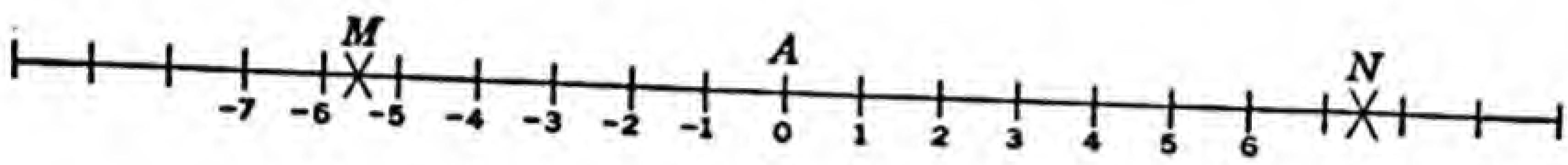


FIG. 1

We say that a is *greater than* b, or that b is *less than* a in case $(a - b)$ is *positive*. We use the signs $>$ and $<$ to indicate *greater than* and *less than*, respectively. We read $M < N$ as "*M is less than N.*"

ILLUSTRATION 1. $-8 > -36$ because $[-8 - (-36)] = 28$.
$2 < 15$. $\quad -5 < 2$. $\quad 0 < 17$. $\quad -15 < 0$.

To say that $M < 0$ is equivalent to saying that M is *negative*. To say that $M > 0$ means that M is *positive*.

Suppose that M and N are two numbers on the scale in Figure 1. Then, to say that $M < N$ means, geometrically, that M is to the *left* of N in Figure 1.

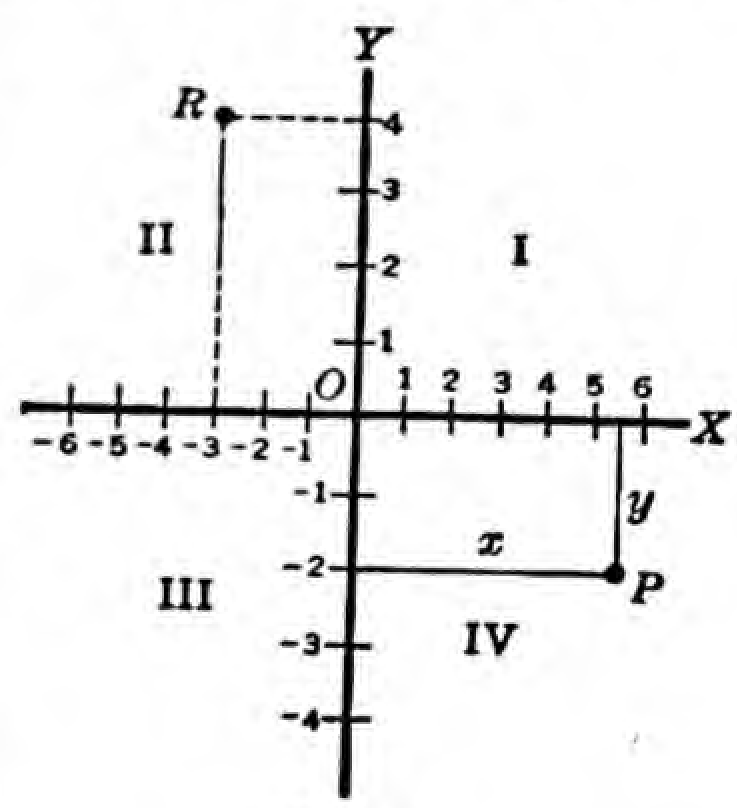


FIG. 2

56. Rectangular coordinates. On each of the perpendicular axes OX and OY in Figure 2, we lay off a scale with O as the zero point on both scales. In the plane of OX and OY we shall measure vertical distances in terms of the unit on OY and horizontal distances in terms of the unit on OX. We agree that horizontal distances will be considered *positive* if measured to the *right* and *negative* if to the *left;* vertical distances will be considered *positive* if measured *upward* and *negative* if *downward.* Let P be any point in the plane.

The *horizontal coordinate*, or the **abscissa** of P, is the perpendicular distance, x, *from* OY *to* P; this directed distance is *positive* if P is to the *right* of OY and *negative* if P is to the *left* of OY.

The *vertical coordinate*, or the **ordinate** of P, is the perpendicular distance, y, *from* OX *to* P; this directed distance is *positive* if P is *above* OX and *negative* if P is *below* OX.

Each of the lines OX and OY is called a **coordinate axis,** and the abscissa and ordinate of P together are called the *rectangular coordinates* of P. The point O at which the axes intersect is called the **origin** of the coordinate system. When the axes are labeled OX and OY as in Figure 2, we sometimes refer to the abscissa as the x-coordinate and to the ordinate as the y-coordinate.

Notice that there is no necessity for using the same unit of length for the scales on OX and OY.

ILLUSTRATION 1. In Figure 2, the coordinates of P are $x = 5\frac{1}{2}$ and $y = -2$. The coordinates of a point are usually written together within parentheses with the abscissa first. Thus, we say that P is the point $(5\frac{1}{2}, -2)$.

Note 1. The coordinate axes divide the plane into four parts called **quadrants,** which we number I, II, III, and IV, *counterclockwise.*

To plot a point, whose coordinates are given, means to locate the point and to mark it with a dot or a cross.

Note 2. The word *line* in this book will refer to a *straight* line unless otherwise specified.

EXERCISE 15

Insert the proper sign, $<$ or $>$, between the numbers.

1. 2 and 5. **2.** -12 and 3. **3.** -3 and 0. **4.** -3 and -5.

Plot the following points on a coordinate system on cross-section paper.

5. $(3, 4)$. **7.** $(1, -2)$. **9.** $(-3, -5)$. **11.** $(0, -2)$.

6. $(3, 0)$. **8.** $(-3, 5)$. **10.** $(-5, 0)$. **12.** $(-8, -1)$.

57. Constants and variables. In a given problem, a *constant* is a number symbol whose value does not change during the discussion. A *variable* is a number symbol which may take on different values. When desirable, we may think of a constant as a variable which may assume only one value.

ILLUSTRATION 1. The volume V of a sphere is given by the formula $V = \frac{4}{3}\pi r^3$ where r is the radius. In considering all possible spheres, π is a constant, approximately 3.1416, but r and V are variables.

58. Functions. If a first variable, x, and a second variable, y, are so related that, whenever a value is assigned to x, a corresponding value (or corresponding values) of y can be determined, we say that y is a *function* of x. Then x is called the *independent variable* and the second variable, y, is called the *dependent variable.* To say that y is a function of x means that *the value of y depends on the value of x.* Any formula in a variable x represents a function of x; the values of the function can be computed from its formula.

ILLUSTRATION 1. In the formula $A = \pi r^2$ for the area of a circle, if r is a variable then A is a variable and A is a function of r.

ILLUSTRATION 2. $(3x^2 + 7x + 5)$ is a function of x. If $x = 2$, the value of the function is $(12 + 14 + 5)$ or 31.

Note 1. If just *one* value of y corresponds to each value of x, we say that y is a *single-valued* function of x; if just *two* values of y correspond to each value of x, then y is a *two-valued* function of x; etc.

59. Graph of a function. Let y represent any function of x. Then, each pair of corresponding values of x and y can be taken as the coordinates of a point in an (x, y) coordinate system. This leads us to adopt the following terminology.

DEFINITION I. *The graph of a function, y, of x is the set of all points (or the* **locus** *of points) whose coordinates form pairs of corresponding values of x and y.*

To graph a function will mean to draw its graph. In graphing a function, we usually plot the values of the independent variable on the horizontal axis of the coordinate system.

ILLUSTRATION 1. If x is the independent variable, in order to graph the function $(\frac{3}{5}x - 3)$, we introduce y to represent it. That is, we let $y = \frac{3}{5}x - 3$. If $x = -5$, then $y = \frac{3}{5}(-5) - 3 = -6$. Hence, one point on the graph is $(-5, -6)$. Similarly, we let $x = 0, -2$, etc., and compute the corresponding values of y given in the following table. We plot $(-5, -6)$, $(-2, -4\frac{1}{5})$, etc., in Figure 3 and join them by a straight line, which is the graph of the function. From the graph, we read that the function equals -2 when $x = 1\frac{2}{3}$, approximately.

$x =$	-5	-2	0	3	6
$y =$	-6	$-4\frac{1}{5}$	-3	$-1\frac{1}{5}$	$\frac{3}{5}$

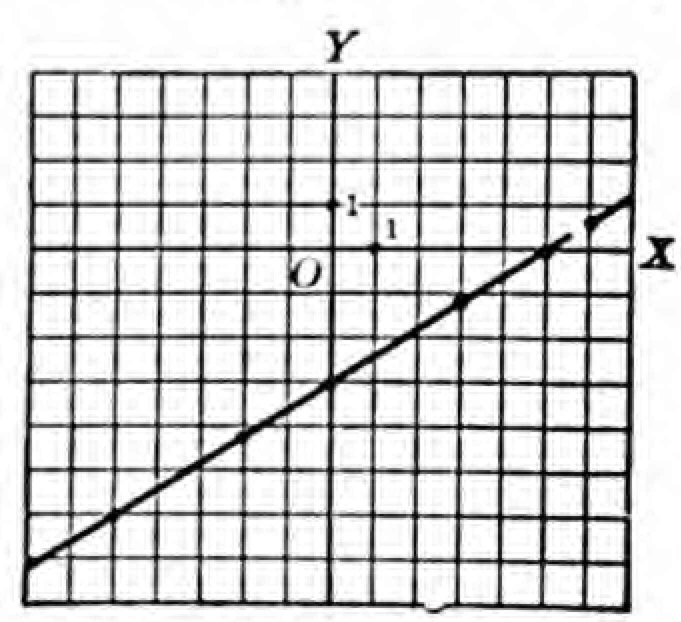


FIG. 3

A **linear function** of x is a polynomial of the *first* degree in x and hence has the form $ax + b$, where a and b are constants. In Illustration 1 we met a special case of the fact that **the graph of a linear function of x is a straight line.** This fact, whose proof we omit, is the basis for the name *linear* function of x. If y is a linear function of x, we need only *two* pairs of values of x and y to obtain the graph, because a straight line is definitely located if we know two points on it. However, in graphing any linear function, we shall compute *three* values of the function in order to check the arithmetic involved.

If a function of x is defined by a formula, in general its graph is a *smooth curve.** To graph such a function, we introduce some letter, such as y, to represent the function, compute a table of corresponding values of x and y, and draw a smooth curve through the corresponding points on a coordinate system.

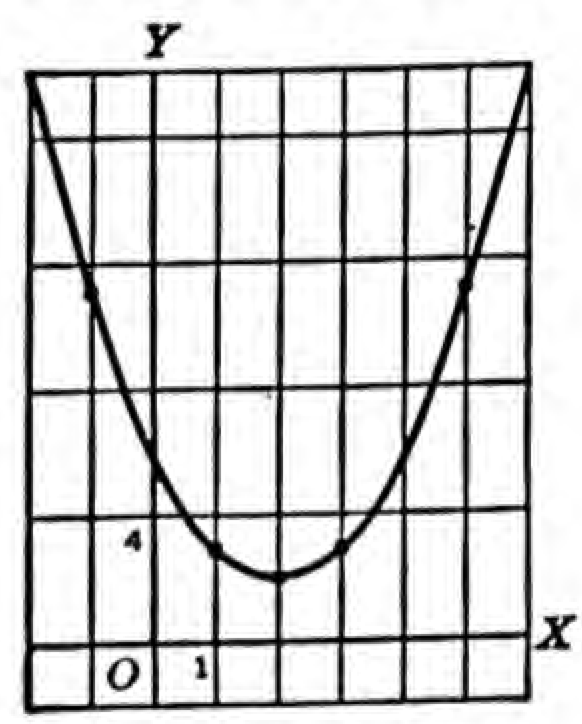


FIG. 4

ILLUSTRATION 2. To graph $x^2 - 4x + 6$, we let

$$y = x^2 - 4x + 6,$$

compute the following table of values, and plot the points. The graph, in Figure 4, is a curve called a **parabola.**

$x =$	-1	1	2	3	5
$y =$	11	3	2	3	11

60. Functions not defined by formulas arise frequently. Sometimes the only information concerning a function consists of a table of corresponding values of the function and the independent variable, where the table may be obtainable by experimental means. In drawing the graph of such a function, sketch a smooth curve through the points obtained, unless otherwise directed. Instead of drawing a smooth curve through the points, we sometimes connect them by segments of straight lines and thus obtain a *broken-line graph.*

EXERCISE 16

The letter x represents an independent variable in all problems. Clearly indicate the scale on each coordinate axis employed.

1. Graph $(x^2 - 6x + 7)$ by computing its values for the following values of x: -1, 0, 2, 3, 4, 6, and 7. From the graph, (*a*) read the values of the function when $x = 5$ and $x = 1$; (*b*) read the values of x for which the function equals 0 or 10.

* Or, in some cases, two or more *disconnected* smooth curves.

Graph the function of x and, from the graph, read the value of x for which the function equals zero.

2. $3x + 5$.
3. $-4 - 3x$.
4. $3 - 4x$.
5. -2.
6. $3x$.
7. 0.
8. $-2x$.
9. $7 - x$.

HINT for Problem 5. Any constant may be considered as a function of any variable x. The graph is parallel to the x-axis.

10. The velocity of sound in air depends on the temperature of the air. By use of the following data, graph the velocity as a function of the temperature. Let the origin of the axes represent 1000 feet on the vertical axis. From the graph, read the velocity if the temperature is 35°; 8.5°; 120°.

VELOCITY, FT. PER SEC.	1030	1040	1060	1080	1110	1140	1170
TEMP. (FAHRENHEIT)	− 30°	− 20°	0°	20°	50°	80°	110°

11. Graph the Centigrade reading C as a function of the Fahrenheit reading F from $-40°$ to $200°$ Fahrenheit by use of $F = \frac{9}{5}C + 32$. From the graph read the temperature in degrees Fahrenheit at 29° Centigrade.

61. Functional notation. Sometimes we represent functions by symbols like $f(x)$, $H(x)$, $K(s)$, etc. The letter in parentheses tells what the independent variable is.

ILLUSTRATION 1. We read "$f(x)$" as "the f-function of x," or for short "f of x." We may represent $3x^2 - 5$ by $f(x)$ and write $f(x) = 3x^2 - 5$; we read this "f of x is $3x^2 - 5$." $H(y)$ would represent a function of y.

If $F(x)$ is any function of x and a is any value of x, then

$F(a)$ represents the value of $F(x)$ when $x = a$.

ILLUSTRATION 2. "$F(a)$" is read "F of a." If $F(x) = 3x^2 - 5 - x$,

$$F(3) = 3 \cdot 3^2 - 5 - 3 = 19;$$
$$F(-3) = 3(-3)^2 - 5 + 3 = 25;$$
$$F(-b^2) = 3(-b^2)^2 - 5 - (-b^2) = 3b^4 - 5 + b^2;$$
$$[F(-2)]^2 = (12 - 5 + 2)^2 = 81.$$

A variable z is said to be a function of *two* variables x and y in case a value of z can be determined corresponding to each pair of values of x and y. Similarly, we may speak of a function of three variables, or of any number of variables.

ILLUSTRATION 3. $F(x, y)$ would be read "F of x and y" and would represent a function of the independent variables x and y. Thus, we may let $F(x, y) = x + 3y^2 + 2$. Then, $F(2, 1) = 2 + 3 + 2 = 7$.

EXERCISE 17

1. If $f(x) = 2x + 3$, find $f(2)$; $f(-3)$; $f(-\frac{1}{4})$; $f(\frac{3}{5})$.
2. If $G(z) = 2z - 3z^2$, find $G(-3)$; $G(5)$; $G(b)$; $G(3x)$.
3. If $F(x) = x^2 - x + 3$, find $F(-2)$; $F(b^2)$; $F(c/d)$; $F(2x - 1)$.
4. If $G(w) = \dfrac{3w + 2}{w}$, find $3G(5)$; $[G(2)]^2$; $G(x^2)$; $\dfrac{G(2)}{G(3)}$.
5. If $K(z) = \dfrac{z + 2}{1 - z}$, find $3K(2)$; $K(s)K(-1)$; $K(c^2)$; $K\left(\dfrac{3x}{y}\right)$; $\dfrac{K(a)}{K(b)}$.
6. If $f(x) = x^3 - 12x + 3$, graph $f(x)$ by use of $f(-4)$, $f(-3)$, $f(-2)$, $f(-1)$, $f(0)$, $f(1)$, $f(2)$, $f(3)$, and $f(4)$.
7. If $F(x, y) = 3x + 2y$, find $F(2, -3)$; $F(a, 2b)$.
8. If $F(x, y) = x^2 + 3xy$, find $F(-1, 2)$; $F(x, 2c)$.

62. Functions defined by equations. A **solution** of an equation in two variables x and y is a **pair of corresponding values** of x and y which satisfy the equation. Usually, an equation in two variables has *infinitely many* solutions.

ILLUSTRATION 1. Consider $3x - 5y = 15$. If $x = 3$, then $9 - 5y = 15$, or $y = -\frac{6}{5}$. Hence, $(x = 3, y = -\frac{6}{5})$ is a solution of the given equation. If $y = 0$, then $3x = 15$ or $x = 5$; hence, $(x = 5, y = 0)$ is another solution. Thus, we could find as many solutions as we might desire.

In case x and y are related by an equation, then usually we may think of y as a function of x and, likewise, of x as a function of y. This is true because, in general, for each value of either variable we can find corresponding values of the other variable by use of the equation. In particular, a *linear* equation in x and y defines either variable as a *linear* function of the other variable.

ILLUSTRATION 2. From $3x - 5y = 15$, $x = 5 + \frac{5}{3}y$ and $y = \frac{3}{5}x - 3$.

63. The graph, or the locus, of an equation in two variables x and y is the locus of all points whose coordinates (x, y) form solutions of the equation. If we think of x as an independent variable, the graph of the equation is *identical with the graph of the function, y, of x, defined by the equation.* In particular, if a, b, and c are constants, **the graph of the linear equation $ax + by = c$ is a straight line.** For, the graph of this equation is the graph of the linear function of x, or of the linear function of y, defined by the equation.

ILLUSTRATION 1. From $3x - 5y = 15$, we obtain $y = \frac{3}{5}x - 3$. The graph of $3x - 5y = 15$ is the graph of the linear function $\frac{3}{5}x - 3$ (page 49).

The abscissa of any point where a graph on an (x, y) coordinate system meets the x-axis is called an **x-intercept** of the graph. The ordinate of any point where the graph meets the y-axis is called a **y-intercept.** To find the *x-intercept* (or *intercepts*) of the graph of an equation in x and y, place $y = 0$ in the equation and solve for x; to find the *y-intercept* (or *intercepts*), place $x = 0$ and solve for y.

To graph a linear equation in x and y, find the solutions corresponding to the x and y intercepts, and one other solution of the equation; then draw the line through the points thus found.

ILLUSTRATION 2. To graph $3x - 5y = 15$, first let $x = 0$ and obtain $0 - 5y = 15$, or $y = -3$; hence, $(0, -3)$ is a point on the graph. If $y = 0$ then $3x - 0 = 15$, or $x = 5$; the x-intercept is 5, or $(5, 0)$ is a point on the graph. These points check with Figure 3, page 49.

64. Equation of a line. An *equation of a curve* on an (x, y) coordinate plane is an equation in the variables x and y *whose graph is the given curve.* If two equations have the same graph, in general the equations differ only in nonessential features. Hence, although a given curve may have infinitely many different equations, we shall refer to any one of these as *the* equation of the curve.

We shall assume without proof the fact that the equation of any straight line on an (x, y) coordinate plane is of the form $ax + by = c$ where a, b, and c are constants. The equation of a line is a linear relation between x and y which is true when and only when the point (x, y) is on the line.

ILLUSTRATION 1. $3x + 2y = 7$ is the equation of a certain straight line. This line also is the graph of $6x + 4y = 14$ because these two equations have the same solutions.

Frequently we refer to a function of a variable x, or to an equation in x and y, by giving the function or equation the name of its graph.

ILLUSTRATION 2. Thus, we may refer to the *line* $3x + 2y = 7$.

ILLUSTRATION 3. The equation of the vertical line 3 units to the left of the y-axis is $x = -3$.

ILLUSTRATION 4. Let P, with coordinates (x, y), be any point on the line through $(0, 0)$ and $(1, 2)$. Then, from similar right triangles in Figure 5,

$$\frac{y}{x} = \frac{2}{1} \quad \text{or} \quad y = 2x;$$

this is the equation of the line.

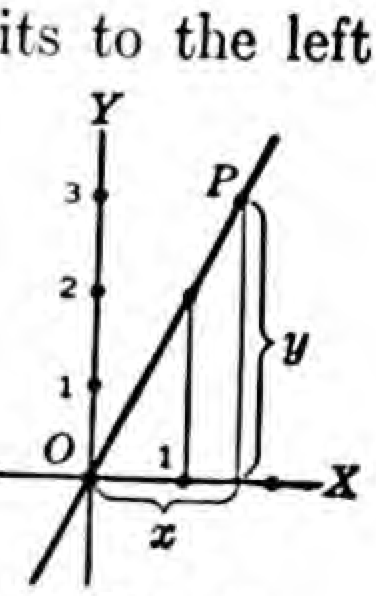


FIG. 5

EXERCISE 18

Graph each equation.

1. $3x + 2y = 6$.
2. $3x - y = 9$.
3. $x + 5y = 15$.
4. $3y - 4x - 12 = 0$.
5. $3x = 5 + 6y$.
6. $4x - 5y = 20$.
7. $3x - 7y = 0$.
8. $2x + 3y = 0$.
9. $x = 7$.
10. $y = -8$.
11. $x = y$.
12. $x = -y$.
13. $5x + 9 = 0$.

14. Find an expression (*a*) for the linear function of y defined by the equation $2x + 7y = 9$; (*b*) for the linear function of x defined by the equation.

15. Give the equation of the horizontal line which lies (*a*) 6 units above OX; (*b*) 4 units below OX.

16. Give the equation of the vertical line which is (*a*) 4 units to the left of OY; (*b*) 7 units to the right of OY.

SUPPLEMENTARY PROBLEMS

Find the equation of the line through the points by use of similar triangles.

17. (0, 0) and (1, 1).
18. (0, 0) and (− 1, − 4).
19. (0, 2) and (3, 5).
20. (− 3, 0) and (1, 6).

Note 1. On an (x, y) coordinate system, on any line which is *not vertical*, select any two points P and Q. Then, the **slope of the line** is defined as *the ratio of the difference of the ordinates of P and Q to the difference of their abscissas.* Thus, two lines are parallel if and only if they have the same slope.

21. If the coordinates of P are (x_1, y_1) and of Q are (x_2, y_2), prove that the slope of line PQ is $(y_2 - y_1)/(x_2 - x_1)$.

Draw the line through the points and find its slope.

22. (3, 5); (5, 9).
23. (2, 4); (6, 2).
24. (− 2, 3); (3, 7).

25. Find the slope of the line $y = 3x + 5$ by first finding the coordinates of two points on the line.

26. Prove that the slope of the line $y = mx + b$ is m and the y-intercept is b. (We call $y = mx + b$ the **slope-intercept form** of the equation of the line.)

Find the slope and the y-intercept of the graph of the equation by first reducing it to the slope-intercept form. Graph the equation.

27. $3x + 2y = 9$.
28. $3x - 4y = 8$.
29. $3x + 8y = 0$.

Determine whether or not the graphs of the equations are parallel by finding their slopes.

30. $\begin{cases} 2x - 5y = 7, \\ 4x - 10y = 5. \end{cases}$

31. $\begin{cases} 3x + 4y = 7, \\ 9x + 12y = 5. \end{cases}$

32. $\begin{cases} 5x - 3y = 2, \\ 10x + 4y = 7. \end{cases}$

CHAPTER FOUR

Systems of Linear Equations

65. A solution of a system of two equations in two unknowns, x and y, is a *pair* of corresponding values of x and y which satisfy both equations. If a system has a solution, the equations are called **simultaneous.**

EXAMPLE 1. Solve graphically:
$$\begin{cases} x - y = 5, & (1) \\ x + 2y = 2. & (2) \end{cases}$$

SOLUTION. 1. In Figure 6, AB is the graph of (1) and CD is the graph of (2). The point of intersection, E, of AB and CD is the only point whose coordinates satisfy both equations.

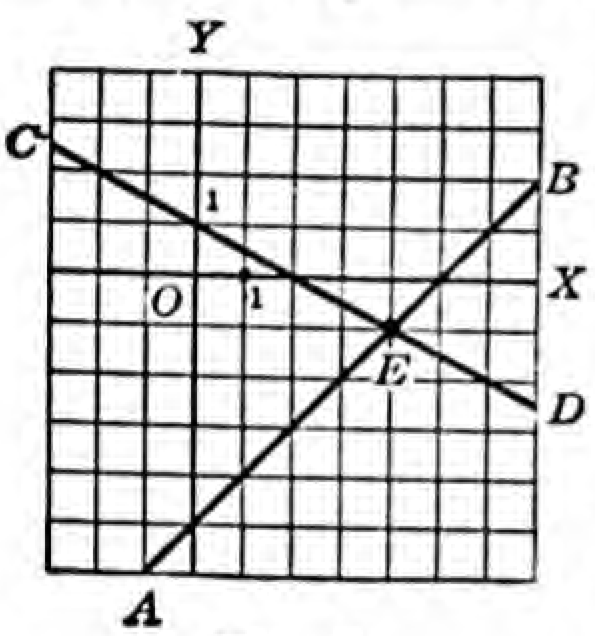


FIG. 6

2. We observe that E has the coordinates $(4, -1)$. Hence, $(x = 4, y = -1)$ is the only solution of the system.

Comment. Usually a graphical solution gives only approximate results, because in obtaining them we estimate the values of certain coordinates by measuring line segments in the graph.

To solve a system of two equations in two unknowns graphically.

1. *Draw the graphs of the equations on one coordinate system.*
2. *Measure the coordinates of any point of intersection of the graphs; these coordinates form a solution of the system.*

Usually a system of two linear equations in two unknowns has just one solution, as was the case in Example 1, but the following special cases may occur.

A. *If the graphs of the equations are parallel lines, the system has no solution and the equations are called* **inconsistent** *equations.*

B. *If the graphs of the equations are the same line, each solution of either equation is also a solution of the other and hence the system has infinitely many solutions. In this case the equations are said to be* **dependent** *equations.*

66. Solution of a system by elimination.

EXAMPLE 1. Solve for x and y:

$$\begin{cases} 4x + 5y = 6, & (1) \\ 2x + 3y = 4. & (2) \end{cases}$$

FIRST SOLUTION. *Elimination by substitution.*

1. Solve (2) for x: $\quad 2x = 4 - 3y; \quad x = \frac{1}{2}(4 - 3y). \quad (3)$

2. Substitute (3) in (1): $\quad 4[\frac{1}{2}(4 - 3y)] + 5y = 6. \quad (4)$

In obtaining equation 4, we have eliminated x by substituting for x from one given equation into the other.

3. Solve (4) for y: $\quad 8 - 6y + 5y = 6; \quad y = 2.$

4. Substitute $y = 2$ in (3): $\quad x = \frac{1}{2}(4 - 6) = -1.$

Hence, the solution of the system is $(x = -1, y = 2)$.

SECOND SOLUTION. *Elimination by addition or subtraction.*

1. Multiply (1) by 3: $\quad 12x + 15y = 18. \quad (5)$

2. Multiply (2) by 5: $\quad 10x + 15y = 20. \quad (6)$

3. Subtract, (5) − (6): $\quad 2x = -2; \quad x = -1. \quad (7)$

In obtaining equation 7, we have eliminated y by *subtraction.*

4. On substituting $x = -1$ in (2) we obtain $\quad 3y = 4 + 2 \quad$ or $\quad y = 2.$

EXAMPLE 2. Solve for x and y:

$$\begin{cases} ax + by = e, & (8) \\ cx + dy = f. & (9) \end{cases}$$

SOLUTION. 1. Multiply (8) by d: $\quad adx + bdy = de. \quad (10)$

2. Multiply (9) by b: $\quad bcx + bdy = bf. \quad (11)$

3. Subtract, (10) − (11): $\quad adx - bcx = de - bf. \quad (12)$

4. Factor left member: $\quad x(ad - bc) = de - bf. \quad (13)$

5. Suppose that $ad - bc \neq 0$ and divide by $ad - bc$ in (13): $\quad x = \dfrac{de - bf}{ad - bc}. \quad (14)$

6. By similar steps [multiplying (8) by c and (9) by a] we find y: $\quad y = \dfrac{af - ce}{ad - bc}. \quad (15)$

Comment. Notice that, in a system with literal coefficients, it is usually best to solve for each unknown in turn by addition or subtraction.

EXERCISE 19

Solve graphically and also by elimination by substitution.

1. $\begin{cases} 3x - y = 7, \\ 2x + 3y = 12. \end{cases}$ 3. $\begin{cases} 3x - 4y = 9, \\ 2x - 3y = 7. \end{cases}$ 5. $\begin{cases} 2x - 3y = 12, \\ 5x + 2y + 8 = 0. \end{cases}$

2. $\begin{cases} y - 2x = 6, \\ x + 2y = 2. \end{cases}$ 4. $\begin{cases} 5y - 2x = 0, \\ 3x + 2y = 0. \end{cases}$ 6. $\begin{cases} 6y - 3x = 10, \\ 9y - 6x = 14. \end{cases}$

Solve graphically and also by elimination by addition or subtraction.

7. $\begin{cases} 3x - 2y = 2, \\ 4y - 3x = 2. \end{cases}$ **9.** $\begin{cases} 5y + 3x = 0, \\ 3y - 2x = 0. \end{cases}$ **11.** $\begin{cases} 4y + 6x = -9, \\ 6y + 4x = -1. \end{cases}$

8. $\begin{cases} 2x - 3y = 9, \\ 4x + 3y = 15. \end{cases}$ **10.** $\begin{cases} 6y - 2x = 1, \\ 9y - 4x = 1. \end{cases}$ **12.** $\begin{cases} 6x + 9y = -14, \\ 3x - 6y = 14. \end{cases}$

Solve for x and y, for r and s, or for w and z by any method.

13. $\begin{cases} 4x - 8y = -3, \\ 11x + 5y = -15. \end{cases}$ **15.** $\begin{cases} 6x - 5y = 3, \\ 4y - 9x = 5. \end{cases}$ **17.** $\begin{cases} 4x + 3y = 6.4, \\ 3x - .5y = 1.5. \end{cases}$

14. $\begin{cases} 2x - y = 8, \\ 7x + 4y = 43. \end{cases}$ **16.** $\begin{cases} 9r + 14s = -\frac{11}{2}, \\ 6r + 21s = -7. \end{cases}$ **18.** $\begin{cases} 5y - 6x = 3.45, \\ 4y + 5x = -.67. \end{cases}$

19. $\begin{cases} \dfrac{2x}{5} + \dfrac{5y}{6} + \dfrac{1}{2} = 0, \\ \dfrac{x}{6} - \dfrac{5y}{9} - \dfrac{5}{2} = 0. \end{cases}$ **20.** $\begin{cases} \dfrac{3x}{2} + \dfrac{4y}{3} + 1 = 0, \\ \dfrac{2x}{3} + \dfrac{y}{4} = \dfrac{7}{12}. \end{cases}$

21. $\begin{cases} 3hx + y = h, \\ 2kx - 3y = k. \end{cases}$ **22.** $\begin{cases} 2x - by = c, \\ 3x + 2by = 4c. \end{cases}$ **23.** $\begin{cases} ax - by = 3, \\ bx - ay = 2. \end{cases}$

24. $\begin{cases} ax + by = 3, \\ bx + ay = 3. \end{cases}$ **25.** $\begin{cases} bw - az - b^2 = 0, \\ aw + bz - bw = a^2. \end{cases}$

26. $\begin{cases} \dfrac{x-2}{x+1} = \dfrac{2+y}{1+y}, \\ \dfrac{5+y}{4+y} - \dfrac{x-3}{x-4} = 0. \end{cases}$ **27.** $\begin{cases} \dfrac{x}{c} + \dfrac{2y}{d} = \dfrac{1}{c} + \dfrac{1}{d}, \\ x - 2y = \dfrac{c^2 - d^2}{cd}. \end{cases}$

Note 1. If the two equations are inconsistent, or dependent, then, in eliminating one unknown, the other is also eliminated; (*a*) if the equations are dependent, an identity $0 = 0$ results; (*b*) if the equations are inconsistent, a contradictory equation, such as $0 = 36$, is obtained.

Proceed with the solution until you recognize that the equations are inconsistent or dependent; check by graphing the equations.

28. $\begin{cases} x - y + 3 = 0, \\ 2x - 2y = 7. \end{cases}$ **30.** $\begin{cases} x + y + 5 = 0, \\ 2x + 2y = -10. \end{cases}$ **32.** $\begin{cases} x - 3y = 5, \\ 2x - 6y = 3. \end{cases}$

29. $\begin{cases} 3x + y = 8, \\ 6x + 2y = 20. \end{cases}$ **31.** $\begin{cases} 3x - 3y = 7, \\ 6x = 14 + 6y. \end{cases}$ **33.** $\begin{cases} 3x + 4y = 8, \\ 6x + 8y = 16. \end{cases}$

Solve by introducing two unknowns.

34. Workmen A and B complete a certain job if they work together for 6 days or if A alone works for 3 days and B alone works for 10 days. How long does it take each man to complete the job alone?

35. A weight of 5 pounds is 6 feet from the fulcrum on the right side of a lever. The lever is balanced if we place a first weight 4 feet from the fulcrum on the right and a second weight 7 feet from the fulcrum on the left, or if we place the first weight 8 feet to the right and the second weight 9 feet to the left of the fulcrum. Find the unknown weights.

36. An airplane, flying with the wind, took 2.5 hours for a 625-mile run and took 4 hours and 10 minutes to return against the same wind. Find the velocity of the wind and the speed of the airplane in calm air.

37. How much silver and lead should be added to 100 pounds of a mixture containing 15% silver and 30% lead to obtain an alloy containing 25% silver and 50% lead?

38. An army messenger will travel at a speed of 60 miles per hour on land, and in a motorboat whose speed is 20 miles per hour in still water. In delivering a message he will go by land to a dock on a river and then on the river against a current of 4 miles per hour. If he reaches his destination in $4\frac{1}{2}$ hours and then returns to his starting point in $3\frac{1}{2}$ hours, how far did he travel by land and how far by water?

39. If a two-digit number is divided by its units' digit, the quotient is 16. If the digits of the given number are reversed, the new number is 18 less than the original one. Find this number.

40. It takes a man 3 hours to row 20 miles downstream on a river and 10 hours to return. Find the rate of the current and the rate at which the man can row in still water.

67. A system of three linear equations in three unknowns usually has one and only one solution. In special cases, however, such a system may have no solution, in which case the equations are called *inconsistent*, or infinitely many solutions, in which case the equations are called *dependent*. Such cases will not be considered in this chapter.

EXAMPLE 1. Solve for x, y, and z:
$$\begin{cases} 3x + y - z = 11, & (1)\\ x + 3y - z = 13, & (2)\\ x + y - 3z = 11. & (3)\end{cases}$$

SOLUTION. 1. Subtract, (1) − (2): $$2x - 2y = -2. \quad (4)$$
Multiply (2) by 3: $$3x + 9y - 3z = 39. \quad (5)$$
Subtract, (5) − (3): $$2x + 8y = 28. \quad (6)$$
2. Solve (4) and (6) for x and y:
Subtract, (6) − (4): $$10y = 30; \quad y = 3.$$
Substitute $y = 3$ in (4): $$2x - 6 = -2; \quad x = 2.$$
3. Substitute $(x = 2, y = 3)$ in (1): $$6 + 3 - z = 11; \quad z = -2.$$
The solution of the given system is $(x = 2, y = 3, z = -2)$.

METHOD I. *Solution of a system of three linear equations in three unknowns.*

1. *From one pair of the equations, eliminate one of the unknowns; eliminate this unknown from another pair of the original equations.*
2. *Solve the resulting equations for the two unknowns in them.*
3. *Substitute the values of the unknowns found in Step 2 in the simplest of the given equations and solve for the third unknown.*

EXERCISE 20

Solve. Do not commence by clearing of fractions.

1. $\begin{cases} 3y - 5x = 1, \\ 3x + z = 1, \\ z + 2y = 2. \end{cases}$ **2.** $\begin{cases} 2x + y = 2, \\ 2y - 5z = 7, \\ 6x + 2z = 1. \end{cases}$ **3.** $\begin{cases} 9x + 7 = 5z, \\ 9x - 5y = 3, \\ 3y + 3z = 2. \end{cases}$

4. $\begin{cases} x - y + 6z = 7, \\ 2x + 3y + 6z = 0, \\ x + 2y + 9z = 3. \end{cases}$ **6.** $\begin{cases} 2A - 3B - C = 0, \\ 2A + 3B - 2C + 1 = 0, \\ A + 3B + 2C = 1. \end{cases}$

5. $\begin{cases} 2x - y + 2z = 2, \\ 12x + y - 3z = 3, \\ 6x - y + 6z = 12. \end{cases}$ **7.** $\begin{cases} 9x + 4y + 3z = 3, \\ y - 6x - 6z + 3 = 0, \\ z - y - x = 2. \end{cases}$

8. $\begin{cases} \dfrac{2}{x} + \dfrac{5}{y} = 5, \\ \dfrac{8}{x} - \dfrac{15}{y} = 6. \end{cases}$ **9.** $\begin{cases} \dfrac{9}{x} + \dfrac{6}{y} = 14, \\ \dfrac{5}{x} + \dfrac{9}{y} = 4. \end{cases}$ **10.** $\begin{cases} x + 2y + u = 4, \\ x + 2z - 3u = 6, \\ 3y - x - 6z = -2, \\ 3u + y - z = -2. \end{cases}$

HINT for Problem 8. Eliminate one unknown by addition or subtraction *without* clearing of fractions.

11. $\begin{cases} \dfrac{2}{x} + \dfrac{1}{y} + \dfrac{3}{z} + 10 = 0, \\ \dfrac{2}{x} - \dfrac{2}{y} + \dfrac{1}{z} = 2, \\ \dfrac{6}{x} + \dfrac{2}{y} - \dfrac{2}{z} = 5. \end{cases}$

HINT for Problem 11. Let

$$\frac{1}{x} = u, \quad \frac{1}{y} = v, \quad \text{and} \quad \frac{1}{z} = w$$

in all equations. Then, first solve for u, v, and w. Finally obtain (x, y, z) by use of (u, v, w).

12. A man divides \$10,000 among three investments, at 3%, 4%, and 6% per annum, respectively. His annual income from the first two investments is \$80 less than his income from the third investment and his total income is \$460 per year. Find the amount invested at each rate.

13. In a three-digit number which is 31 times the sum of the digits, the units' digit is one half the sum of the other digits. If the digits are reversed, the new number is 99 greater than the original number. Find its digits.

★68. Solution of two linear equations by determinants. On page 56, we found the solution of the following system:

$$\begin{cases} ax + by = e, & (1) \\ cx + dy = f. & (2) \end{cases}$$

If $ad - bc \neq 0$, $$x = \frac{de - bf}{ad - bc}; \quad y = \frac{af - ce}{ad - bc}. \tag{3}$$

The symmetrical nature of the numerators and denominators in (3) was noticed by early mathematicians and led to the introduction of the following notation. The symbol on the left below is called a **determinant** and is an abbreviation for $ad - bc$. Or, by definition,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \tag{4}$$

We read (4) as follows: "the determinant (a, b), (c, d) equals $ad - bc$." We call a, b, c, and d the **elements,** and $ad - bc$ the **expansion** of the determinant. This determinant is said to be of the *second* order because it has two rows and two columns of elements.

Illustration 1. $\begin{vmatrix} 3 & 2 \\ -4 & -5 \end{vmatrix} = 3(-5) - (-4)(2) = -15 + 8 = -7.$

By the definition of a determinant, we observe that the solution of system [(1), (2)] given in equations (3) can be written as follows:

$$\left(\text{if } \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0\right) \qquad x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}; \qquad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}. \tag{5}$$

We refer to the denominator in (5) as *the determinant of the coefficients of the unknowns.*

Summary. *If the determinant of the coefficients of the unknowns is not zero, the system of linear equations has just one solution, in which the value of each unknown is the quotient of two determinants:*

1. *the denominator determinant, for all unknowns, is formed by writing the coefficients of the unknowns by rows in a regular order;*
2. *the numerator, for any unknown, is obtained from the denominator by replacing the coefficients of this unknown by the constant terms.*

Illustration 2. In (5), the constant terms e and f are in the 1st column in the numerator for x and in the 2d column in the numerator for y.

EXAMPLE 1. Solve by determinants: $\begin{cases} 2x - 4y = -14, \\ 3x + 7y = 5. \end{cases}$

SOLUTION. From (5),

$$x = \frac{\begin{vmatrix} -14 & -4 \\ 5 & 7 \end{vmatrix}}{\begin{vmatrix} 2 & -4 \\ 3 & 7 \end{vmatrix}} = \frac{-78}{26} = -3; \qquad y = \frac{\begin{vmatrix} 2 & -14 \\ 3 & 5 \end{vmatrix}}{\begin{vmatrix} 2 & -4 \\ 3 & 7 \end{vmatrix}} = \frac{52}{26} = 2.$$

★EXERCISE 21

Solve for x and y by use of determinants.

1–17. Solve Problems 1–17 of Exercise 19.

18. $\begin{cases} 3x - 5 = 0, \\ 2x - 3y = 18. \end{cases}$ **19.** $\begin{cases} 3x - 7y = 0, \\ 2x + 5y = 0. \end{cases}$ **20.** $\begin{cases} ax - by = 1, \\ bx - ay = 1. \end{cases}$

21–24. Solve Problems 21–24 of Exercise 19.

25–27. Problems 28–30 of Exercise 19. In each case tell why formulas 5, page 60, cannot be used; then solve graphically.

★69. Determinants of the third order. The symbol

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \tag{1}$$

is called a *determinant of the third order* and is defined as an abbreviation for the expression

$$a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3. \tag{2}$$

The numbers a_1, a_2, a_3, b_1, c_1, etc. are called the *elements* and the polynomial in (2) is called the *expansion* of the determinant.

Note 1. The terms in (2) can be remembered as follows. Repeat the first two columns of the determinant at the right of the determinant as in the diagram. *Find the product of each set of three elements through which an arrow is drawn. Give the product a* **plus** *sign if its arrow points* **downward**, *and a* **minus** *sign if the arrow points* **upward**.

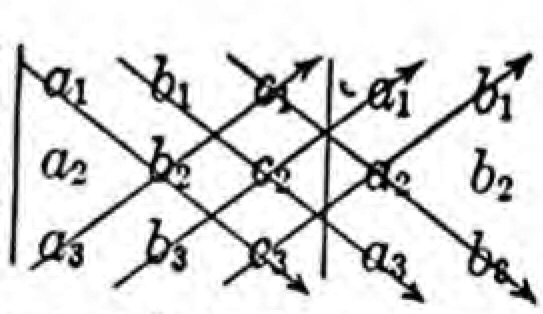
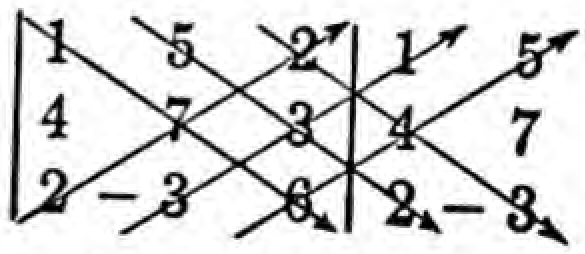


ILLUSTRATION 1. $\begin{vmatrix} 1 & 5 & 2 \\ 4 & 7 & 3 \\ 2 & -3 & 6 \end{vmatrix} =$



$$(1\cdot7\cdot6) + (5\cdot3\cdot2) + [2\cdot4\cdot(-3)] - (2\cdot7\cdot2) - [(-3)\cdot3\cdot1] - (6\cdot4\cdot5)$$
$$= 42 + 30 - 24 - 28 + 9 - 120 = -91.$$

★EXERCISE 22

Obtain the expansion of each determinant.

1. $\begin{vmatrix} 1 & -3 & -2 \\ 2 & 1 & 4 \\ -2 & 1 & -3 \end{vmatrix}$.

2. $\begin{vmatrix} 5 & -3 & 1 \\ 2 & 0 & -2 \\ 1 & 3 & -4 \end{vmatrix}$.

3. $\begin{vmatrix} -2 & 0 & 2 \\ 1 & 3 & -1 \\ 4 & -2 & 3 \end{vmatrix}$.

4. $\begin{vmatrix} 0 & 2 & 1 \\ -1 & 3 & 5 \\ 3 & 4 & -1 \end{vmatrix}$.

5. $\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$.

6. $\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$.

★70. Solution of three linear equations by determinants.

EXAMPLE 1. Solve for x, y, and z:

$$\begin{cases} a_1x + b_1y + c_1z = d_1, & (1) \\ a_2x + b_2y + c_2z = d_2, & (2) \\ a_3x + b_3y + c_3z = d_3. & (3) \end{cases}$$

SOLUTION. 1. By the method of Section 67, we solve the system of equations [(1), (2), (3)] for x, y, and z and obtain the following fractions, provided, of course, that the denominator in them is not zero:

$$x = \frac{d_1b_2c_3 + d_3b_1c_2 + d_2b_3c_1 - d_3b_2c_1 - d_1b_3c_2 - d_2b_1c_3}{a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3}; \tag{4}$$

$$y = \frac{a_1d_2c_3 + a_3d_1c_2 + a_2d_3c_1 - a_3d_2c_1 - a_1d_3c_2 - a_2d_1c_3}{a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3}; \tag{5}$$

$$z = \frac{a_1b_2d_3 + a_3b_1d_2 + a_2b_3d_1 - a_3b_2d_1 - a_1b_3d_2 - a_2b_1d_3}{a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3}. \tag{6}$$

2. The *denominator* in these fractions is seen to be the expansion of the determinant discussed in Section 69. The numerator for x in (4) differs from the denominator only in that the a's *of the denominator are replaced by d's.* Hence, in the following expression for x, the numerator determinant is obtained from the denominator by replacing the column of a's by d's; this numerator was explicitly verified in Problem 5 of Exercise 22. Similarly, we may rewrite (5) and (6) in determinant form and finally obtain

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}; \quad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}; \quad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}. \tag{7}$$

The solution of Example 1 shows that Steps 1 and 2 of the summary on page 60 apply without a change in wording to a system of three linear equations in three unknowns.

Formulas 7 of this section, or formulas 5 of page 60 for a system of two equations, apply when and only when the denominator determinant is *not zero*. In more advanced mathematics it is proved that, in case the denominator determinant is zero, the equations of the system are either dependent or inconsistent.

Note 1. As far as the Western world is concerned, determinants were invented in 1693 by the German mathematician LEIBNIZ (1646–1716). However, determinants were invented at least ten years earlier by SEKI-KOWA (1642–1708), the great Japanese mathematician. Due to the isolation of Japan, the work of Seki-Kowa had no influence on mathematical development outside of Japan.

EXAMPLE 2. Solve by determinants: $\begin{cases} 3x + y - z = 14, \\ x + 3y - z = 16, \\ x + y - 3z = -10. \end{cases}$

SOLUTION. By use of formulas 7,

$$x = \frac{\begin{vmatrix} 14 & 1 & -1 \\ 16 & 3 & -1 \\ -10 & 1 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 1 & 1 & -3 \end{vmatrix}} = \frac{-126 + 10 - 16 - 30 + 14 + 48}{-27 - 1 - 1 + 3 + 3 + 3} = \frac{-100}{-20} = 5.$$

$$y = \frac{\begin{vmatrix} 3 & 14 & -1 \\ 1 & 16 & -1 \\ 1 & -10 & -3 \end{vmatrix}}{-20} = \frac{-120}{-20} = 6. \quad z = \frac{\begin{vmatrix} 3 & 1 & 14 \\ 1 & 3 & 16 \\ 1 & 1 & -10 \end{vmatrix}}{-20} = \frac{-140}{-20} = 7.$$

★EXERCISE 23

1–7. Solve Problems 1 to 7 of Exercise 20 by use of determinants.

Solve for x, y, and z by use of determinants.

8. $\begin{cases} ax = g, \\ bx + cy = h, \\ dx + ey + fz = k. \end{cases}$

9. $\begin{cases} ax + by = c, \\ ax + cz = b, \\ by + cz = a. \end{cases}$

10. Carry through the elimination by addition or subtraction which is necessary to obtain the expression for x in equation 4, page 62.

CHAPTER FIVE

Quadratic Equations

71. Introduction to imaginary numbers. By definition, R is a square root of A if $R^2 = A$. Hence, -1 has no positive or negative number R as a square root because, in such cases, R^2 is *positive* and cannot satisfy $R^2 = -1$. For this reason, we introduce the symbol $\sqrt{-1}$ as a new variety of number, called an **imaginary number,** with the property that $\sqrt{-1}\sqrt{-1} = -1$. And, for contrast, we call positive and negative numbers and zero **real numbers.** For convenience we let $i = \sqrt{-1}$. Then by definition $i^2 = -1$. We shall assume* that the operations of addition, subtraction, multiplication, and division may be applied to combinations of i and real numbers as if i were an ordinary real literal number, with $i^2 = -1$. Then, in particular, $(-i)^2 = i^2 = -1$, so that $-i$, as well as $+i$, is a square root of -1.

ILLUSTRATION 1. $$(3 + 5i)(4 + i) = 12 + 23i + 5i^2$$
$$= 12 + 23i - 5 = 7 + 23i.$$

If P is any positive number, we verify that

$$(i\sqrt{P})^2 = i^2P = -P; \quad (-i\sqrt{P})^2 = i^2P = -P.$$

Hence, the negative number $-P$ has the two square roots $\pm i\sqrt{P}$. Hereafter, we agree that the symbol $\sqrt{-P}$ or $(-P)^{\frac{1}{2}}$ represents the particular square root $i\sqrt{P}$. Then, $-P$ has the two square roots $\pm\sqrt{-P} = \pm i\sqrt{P}$. This agreement about the meaning of $\sqrt{-P}$ is equivalent to saying that *we should proceed as follows in dealing with the square root of a negative number:*

$$\sqrt{-P} = \sqrt{(-1)\cdot P} = \sqrt{-1}\sqrt{P} = i\sqrt{P}. \tag{1}$$

ILLUSTRATION 2. The square roots of -5 are $\pm\sqrt{-5} = \pm i\sqrt{5}$.

$$\sqrt{\frac{-9a^4}{2}} = \sqrt{(-1)\cdot\frac{9a^4}{2}} = \sqrt{-1}\sqrt{\frac{9a^4\cdot 2}{2\cdot 2}} = \frac{3a^2i\sqrt{2}}{2}.$$

ILLUSTRATION 3. $\sqrt{-4}\sqrt{-9} = (i\sqrt{4})(i\sqrt{9}) = 6i^2 = -6.$

* This procedure can be justified by a more advanced discussion.

Note 1. The formula $\sqrt{a}\sqrt{b} = \sqrt{ab}$ was proved only for the case where $\sqrt{a}$ and $\sqrt{b}$ are real. We can verify that the formula does not hold if a and b are negative. Thus, by the formula,

$$\sqrt{-4}\sqrt{-9} = \sqrt{(-4)(-9)} = \sqrt{36} = 6,$$

which is wrong, because the correct result is -6 (in Illustration 3).

If a and b are *real* numbers, we call $(a + bi)$ a **complex number,** whose *real* part is a and *imaginary part* is bi. If $b \neq 0$, we call $(a + bi)$ an **imaginary number.** A **pure imaginary number** is one whose real part is *zero;* that is, $(a + bi)$ is a pure imaginary if $a = 0$ and $b \neq 0$. Any real number a is thought of as a complex number in which the coefficient of the imaginary part is zero; that is, $a = a + 0i$.

ILLUSTRATION 4. $(2 - 3i)$ is an imaginary number. The real number 6 can be thought of as $(6 + 0i)$. Also, $0 = 0 + 0i$.

Note 2. In this book, unless otherwise stated, all literal numbers represent real numbers, except that hereafter i will always represent $\sqrt{-1}$. Any literal number in a radical of *even* order will be supposed *positive,* if this is possible and adds to our convenience.

Positive integral powers of i can be easily computed by recalling that $i^2 = -1$ and hence that $i^4 = 1$.

ILLUSTRATION 5. $i^3 = i^2 i = (-1)i = -i$. $i^{13} = i^{12} i = (i^4)^3 i = 1 \cdot i = i$.

EXERCISE 24

Express by use of the symbol i and simplify the radical.

1. $\sqrt{-25}$. **3.** $\sqrt{-23}$. **5.** $\sqrt{-\frac{1}{9}}$. **7.** $\sqrt{-\frac{3}{2}}$. **9.** $\sqrt{-.09}$

2. $\sqrt{-36}$. **4.** $\sqrt{-28}$. **6.** $\sqrt{-\frac{9}{64}}$. **8.** $\sqrt{-a^2}$. **10.** $\sqrt{-9x^2}$.

11. $\sqrt{-8x^2}$. **12.** $\sqrt{-12xw^3}$. **13.** $\sqrt{-128x^4y^4}$. **14.** $\sqrt{-48a^4y^3}$.

Give the two square roots of each number.

15. -81. **16.** $-\frac{16}{25}$. **17.** $-\frac{9}{16}$. **18.** -63. **19.** $-\frac{4}{7}$.

Perform any indicated operation and simplify by use of $i^2 = -1$.

20. i^5. **21.** i^7. **22.** i^4. **23.** i^6. **24.** i^8. **25.** i^{18}. **26.** i^{29}.

27. $(3 - i)(3 + i)$. **28.** $(3i + 5)(4 - 3i)$. **29.** $(3 + 2i)(3 - 2i)$.

30. $(2i + 3)^2$. **31.** $(4 + 3i)^2$. **32.** $(5 - 2i)^2$. **33.** $(3i - 4)^2$.

34. $(4i - 7i^3)(2i + 5i^2)$. **35.** $(2i + 4i^4 - i^2)(2 + 3i)$.

36. $\sqrt{-2}\sqrt{-8}$. **37.** $\sqrt{-2}(3 - 5\sqrt{-4})$. **38.** $(5 - \sqrt{-8})^2$.

39. If $f(x) = 3x^2 + 2x - 7$, find $f(2i)$; $f(-3i)$; $f(2 - 5i)$.

72. A quadratic equation, or one of the *second degree,* is an integral rational equation in which, after like terms are collected, those of highest degree in the variables are of the *second* degree. A quadratic equation in x can be reduced to the standard form $\boldsymbol{ax^2 + bx + c = 0}$, where a, b, and c are constants and $a \neq 0$. A *complete* quadratic equation in x is one for which $b \neq 0$, and a *pure* quadratic equation is one for which $b = 0$.

To solve a pure quadratic equation in x, we solve the equation for x^2 and extract square roots.

EXAMPLE 1. Solve: $\quad 7y^2 = 18 + 3y^2.$

SOLUTION. $\quad 4y^2 = 18; \quad y^2 = \frac{9}{2}.$

Thus, y must be a square root of $\frac{9}{2}$. Hence, by use of Table I,

$$y = \pm\sqrt{\frac{9}{2}} = \pm\sqrt{\frac{9 \cdot 2}{2 \cdot 2}} = \pm\frac{3}{2}\sqrt{2} = \pm\frac{3}{2}(1.414) = \pm 2.121.$$

EXAMPLE 2. Solve: $\quad 2y^2 + 35 = -5y^2.$

SOLUTION. $7y^2 = -35; \; y^2 = -5.$ Hence, $y = \pm\sqrt{-5} = \pm i\sqrt{5}.$

Note 1. If the coefficients in a quadratic equation are explicit numbers and if a radical occurs in any solution which is a real number, compute the decimal value of the solution by use of Table I. If it is desired to check such a solution, substitute the *radical* form instead of the approximate decimal value, which, as a rule, could not lead to an absolute check.

73. To solve an equation of any degree in x by use of factoring:

1. *Transpose all terms to one member and thus obtain zero as the other member. Factor the first member if possible.*

2. *Place each factor equal to zero and solve for x.*

EXAMPLE 1. Solve: $\quad 6x^2 + 5x = 6.$

SOLUTION. 1. Subtract 6: $\quad 6x^2 + 5x - 6 = 0;$

$$(3x - 2)(2x + 3) = 0.$$

2. The equation is satisfied if $\; 3x - 2 = 0 \;$ or if $\; 2x + 3 = 0.$

3. If $3x - 2 = 0$, then $3x = 2$; $x = \frac{2}{3}$ is one solution.

4. If $2x + 3 = 0$, then $2x = -3$; $x = -\frac{3}{2}$ is a second solution.

EXAMPLE 2. Solve: $\quad 4x^2 + 20x + 25 = 0.$

SOLUTION. 1. Factor: $\; (2x + 5)^2 = 0; \;$ or $\; (2x + 5)(2x + 5) = 0.$

2. If $2x + 5 = 0$, then $x = -\frac{5}{2}$. Since each factor gives the same value for x, we agree to say that the equation has *two equal roots.*

In solving an equation, if both sides are divided by an expression *involving the unknowns*, solutions may be *lost*.

EXAMPLE 3. Solve: $5x^2 = 8x.$

SOLUTION. 1. Transpose $8x$: $5x^2 - 8x = 0;$ $x(5x - 8) = 0.$

2. Hence, $x = 0$ or $5x - 8 = 0$; the solutions are 0 and $\frac{8}{5}$.

INCORRECT SOLUTION. Divide both sides of $5x^2 = 8x$ by x:

$$5x = 8.$$

Then, *incorrectly*, we obtain $x = \frac{8}{5}$ as the *only* solution.

EXERCISE 25

Solve for x.

1. $5x^2 = 125.$
2. $3x^2 = 12.$
3. $x^2 = -9.$
4. $2x^2 = 3.$
5. $4x^2 = -9.$
6. $9x^2 = -25.$
7. $4x^2 = c.$
8. $3ax^2 = h.$
9. $7x^2 = 5 - 3x^2.$
10. $9x^2 + 49 = 0.$
11. $\frac{1}{2}x^2 - 1 = \frac{1}{5}x^2.$
12. $18x^2 + 64 = 0.$
13. $4ax^2 - c = d.$
14. $4a + 2cx^2 = 4d.$

Solve by factoring.

15. $x^2 - 3x = 10.$
16. $y^2 - 5y = 14.$
17. $2x^2 + 5x = 3.$
18. $3x^2 - 2x = 5.$
19. $5x^2 - 9x = 0.$
20. $6x^2 = 15x.$
21. $21x = 14x^2.$
22. $9x^2 - 144 = 0.$
23. $8x^2 + 3 = 10x.$
24. $16x^2 = 24x - 9.$
25. $12 - 5x^2 = 17x.$
26. $25y^2 = 20y - 4.$
27. $8 = 22x - 15x^2.$
28. $4w^2 + 7w = 15.$
29. $8x^2 + 2x = 15.$
30. $7x^2 + 9x = 10.$
31. $9x^2 - 4 = 5x.$
32. $6 + 5x = 6x^2.$

Solve for x or for w.

33. $3bx^2 + cx = 0.$
34. $x^2 + ax - 6a^2 = 0.$
35. $2x^2 + bx - 3b^2 = 0.$
36. $3w^2 - bw = 4b^2.$
37. $2a^2x^2 - abx - 3b^2 = 0.$
38. $6cx^2 + ab = 3ax + 2bcx.$
39. $(x + 3)(2x - 5)(3x + 7) = 0.$
40. $6x^3 + x^2 - 15x = 0.$

74. Completing a square. A binomial $x^2 + px$ becomes a *perfect square* if we add *the square of one half of the coefficient of x*. That is, we complete a square if we add $\left(\frac{p}{2}\right)^2$ or $\frac{p^2}{4}$:

$$x^2 + px + \frac{p^2}{4} = \left(x + \frac{p}{2}\right)^2. \qquad (1)$$

ILLUSTRATION 1. To make $x^2 - 7x$ a perfect square, we add $(\frac{7}{2})^2$ or $\frac{49}{4}$:

$$x^2 - 7x + \tfrac{49}{4} = (x - \tfrac{7}{2})^2.$$

75. To solve a quadratic equation in x by completing a square:

1. *Transpose all terms involving x to the left side and all other terms to the right member and collect terms.*

2. *Divide both members by the coefficient of x^2.*

3. *Complete a square on the left by adding the square of one half of the absolute value of the coefficient of x to both sides.*

4. *Rewrite the left member as the square of a binomial.*

5. *Extract square roots, using the double sign on the right.*

EXAMPLE 1. Solve: $3x^2 - 8x + 2 = 0.$

SOLUTION. 1. $3x^2 - 8x = -2;$ or $x^2 - \frac{8}{3}x = -\frac{2}{3}.$

2. Since $(\frac{8}{3} \div 2) = \frac{4}{3}$, add $(\frac{4}{3})^2$ or $\frac{16}{9}$ to complete a square:

$$x^2 - \tfrac{8}{3}x + (\tfrac{4}{3})^2 = \tfrac{16}{9} - \tfrac{2}{3}; \quad \text{or} \quad (x - \tfrac{4}{3})^2 = \tfrac{10}{9}.$$

3. Extract square roots: $x - \frac{4}{3} = \pm\frac{1}{3}\sqrt{10}$; or $x = \frac{4}{3} \pm \frac{1}{3}\sqrt{10}$;

$$x = \frac{4 + 3.162}{3} = 2.387, \quad \text{and} \quad x = \frac{4 - 3.162}{3} = .279 \qquad \text{(Table I)}$$

EXAMPLE 2. Solve by completing a square: $x^2 + 4x + 7 = 0.$

SOLUTION. 1. $x^2 + 4x = -7.$

2. Since $(4 \div 2) = 2$, we add 2^2 or 4 to both sides:

$$x^2 + 4x + 4 = 4 - 7; \quad \text{or} \quad (x + 2)^2 = -3.$$

3. Hence, $x + 2 = \pm\sqrt{-3} = \pm i\sqrt{3};$

$$x = -2 \pm i\sqrt{3}.$$

EXAMPLE 3. Solve by completing a square: $ax^2 + bx + c = 0.$

SOLUTION. 1. Subtract c: $\quad ax^2 + bx = -c.$

2. Divide by a: $\quad x^2 + \dfrac{b}{a}x = -\dfrac{c}{a}.$

3. Add $\left(\dfrac{b}{2a}\right)^2$, or $\dfrac{b^2}{4a^2}$: $\quad x^2 + \dfrac{b}{a}x + \left(\dfrac{b}{2a}\right)^2 = \dfrac{b^2}{4a^2} - \dfrac{c}{a}.$

4. Simplify: $\quad \left(x + \dfrac{b}{2a}\right)^2 = \dfrac{b^2 - 4ac}{4a^2}.$

5. Extract square roots: the equation $ax^2 + bx + c = 0$ is satisfied when and only when

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}.$$

6. Subtract $\dfrac{b}{2a}$: $\quad x = -\dfrac{b}{2a} \pm \dfrac{\sqrt{b^2 - 4ac}}{2a} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$

76. The quadratic formula. In Example 3, page 68, the equation

$$ax^2 + bx + c = 0 \tag{1}$$

was solved by the method of completing a square; the solutions are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{2}$$

We call (2) the *quadratic formula.* In (2), it is permissible for a, b, and c to have any values, with $a \neq 0$. To solve a quadratic equation by the quadratic formula, first reduce the equation to the *standard form* $ax^2 + bx + c = 0$ and list the values of a, b, and c.

ILLUSTRATION 1. To solve $3x^2 - 6x - 2 = 0$, we observe that $a = 3$, $b = -6$, and $c = -2$. Hence, from the quadratic formula,

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 3 \cdot (-2)}}{6} = \frac{6 \pm 2\sqrt{15}}{6} = \frac{3 \pm 3.873}{3};$$

$$x = \frac{3 + 3.873}{3} = 2.291, \text{ and } x = \frac{3 - 3.873}{3} = -.291.$$

ILLUSTRATION 2. To solve $2x^2 - 4x + 5 = 0$, we notice that $a = 2$, $b = -4$, and $c = 5$. Hence, from the quadratic formula,

$$x = \frac{4 \pm \sqrt{16 - 40}}{4} = \frac{4 \pm \sqrt{-24}}{4} = \frac{4 \pm 2i\sqrt{6}}{4} = \frac{2 \pm i\sqrt{6}}{2}.$$

EXAMPLE 1. Solve for x: $\qquad x^2 - 3ex + 5dx - 15de = 0.$

SOLUTION. 1. Group terms in x: $\qquad x^2 + x(-3e + 5d) - 15de = 0.$

2. In the standard notation, $a = 1$, $b = -3e + 5d$, and $c = -15de$. From the quadratic formula,

$$x = \frac{-(-3e + 5d) \pm \sqrt{(5d - 3e)^2 - 4(-15de)}}{2}.$$

The student should simplify to obtain $x = 3e$ and $x = -5d$.

Note 1. Table I is useful in detecting perfect square numbers.

77. Summary for solution of quadratic equations. A pure quadratic equation should be solved by merely extracting square roots, as in Section 72. Any other quadratic equation should be solved by factoring if factors can be *easily* recognized. In all other cases, solve by use of the quadratic formula, unless otherwise specified. The method of completing the square is *not recommended* in any problem unless specifically requested; this method was introduced mainly as a means for deriving the quadratic formula.

EXERCISE 26

Solve for x by completing a square.

1. $x^2 + 6x - 7 = 0.$
2. $x^2 + 10x + 24 = 0.$
3. $9x^2 + 6x = 1.$
4. $5x^2 - 2 = 4x.$
5. $x^2 + 13 = 6x.$
6. $4x^2 + 13 = 12x.$
7. $4x^2 + 25 = 20x.$
8. $9x^2 + 1 = 12x.$
9. $2x^2 = 4x - 7.$
10. $x^2 - ax = 6a^2.$
11. $2x^2 - 5bx = 3b^2.$
12. $3x^2 + ax - b = 0.$
13. $hx^2 + kx + P = 0.$

Solve for x or y by use of the quadratic formula.

14. $6x^2 + x - 2 = 0.$
15. $3y^2 + 2y - 5 = 0.$
16. $7y^2 - 8y = 12.$
17. $y^2 - 2y + 10 = 0.$
18. $x^2 + 13 = 4x.$
19. $4x^2 + 9 = 12x.$
20. $16x^2 - 25 = 0.$
21. $9x^2 + 6x = 1.$
22. $4 + 4x = 6x^2.$
23. $2x^2 + 3 = 8x.$
24. $4x^2 + 3 = 2x.$
25. $4x^2 + 13 = 4x.$
26. $4x^2 + 13 = 12x.$
27. $18x^2 + 33x = 40.$
28. $21x^2 + 19x = 12.$
29. $6x^2 - 5dx - 6d^2 = 0.$
30. $ax^2 - dx + 3c = 0.$
31. $5ky^2 - 3ky + 6 = 0.$
32. $6hy^2 - 4hy + 10 - 15y = 0.$
33. $3x^2 + 3hx^2 - 6x + 5hx - 10 + 5x = 0.$

Solve for x or y or z by the most convenient method.

34. $y^2 - 33 = 8y.$
35. $16y^2 + 9 = 24y.$
36. $14x^2 - x = 3.$
37. $3z^2 - 6 = 2z.$
38. $49x^2 + 4 = 14x.$
39. $25x^2 - 20x = 1.$
40. $16 - 5x^2 = 0.$
41. $6x^2 = 7x.$
42. $20x^2 + 13x = 21.$
43. $\dfrac{4}{3 - x} + \dfrac{2x}{5 + x} = 1.$
44. $\dfrac{3x}{x - 2} - \dfrac{1}{x^2 - 4} = 2.$
45. $\dfrac{7}{1 + 2y} - \dfrac{5y}{2y^2 + 3y + 1} = \dfrac{1}{3}.$
46. $\dfrac{5 - x}{2 - x} - \dfrac{3 - 2x}{2x} = 1.$
47. $\frac{1}{2}gx^2 + ax = 3S.$
48. $4x^2 + 2bx + b = 1.$
49. $3x^2 + hx + 3kx + hk = 0.$
50. $ax^2 - 2bx = 2x + 3.$
51. Solve for x in terms of y: $2y^2 + 15x^2 - 2 - x + 3y - 13xy = 0.$
52. Solve for y in terms of x in Problem 51.

Solve each problem by introducing only one unknown number.

53. Divide 45 into two parts whose product is 434.

54. The area of a rectangle is 221 square feet and one side is 4 feet longer than the other. Find the dimensions.

55. After plowing a uniform border inside of a rectangular field 50 rods long by 40 rods wide, a farmer finds that he has plowed 60% of the field. Find the width of the border.

56. Find two consecutive integers whose product is 306.

57. After traveling for 50 miles at a certain speed, a motorist increases his speed by 10 miles per hour and travels 100 miles farther. If he took 3 hours to cover the 150 miles, find his speed during the first 50 miles.

58. If an object is shot vertically from the surface of the earth with an initial velocity of v feet per second, and if air resistance and other disturbing factors are neglected, it is proved in physics that $s = vt - \frac{1}{2}gt^2$, where s feet is the height of the object above the surface at the end of t seconds and $g = 32$, approximately. (*a*) Solve for t in terms of s. (*b*) If $v = 200$ feet, use the result of part (*a*) to find when $s = 500$ feet and when the object will hit the ground.

59. If A is the measured cross-section area of a chimney, its so-called effective area E is the smallest root of the equation $E^2 - 2AE + A^2 - .36A = 0$. Solve for E in terms of A, and, from the result, find E if $A = 20$ square feet.

78. Graph of a quadratic function. A *quadratic function* of x is a polynomial of the *second* degree in x and hence has the form

$$ax^2 + bx + c,$$

where a, b, and c are constants and $a \neq 0$.

EXAMPLE 1. Graph the function $x^2 - 2x - 3$.

SOLUTION. Let $y = x^2 - 2x - 3$. We select values for x and compute the corresponding values for y. In Figure 7, we plot the points $(-3, 12)$, $(-2, 5)$, etc. The

$x =$	-3	-2	0	1	2	4	5
$y =$	12	5	-3	-4	-3	5	12

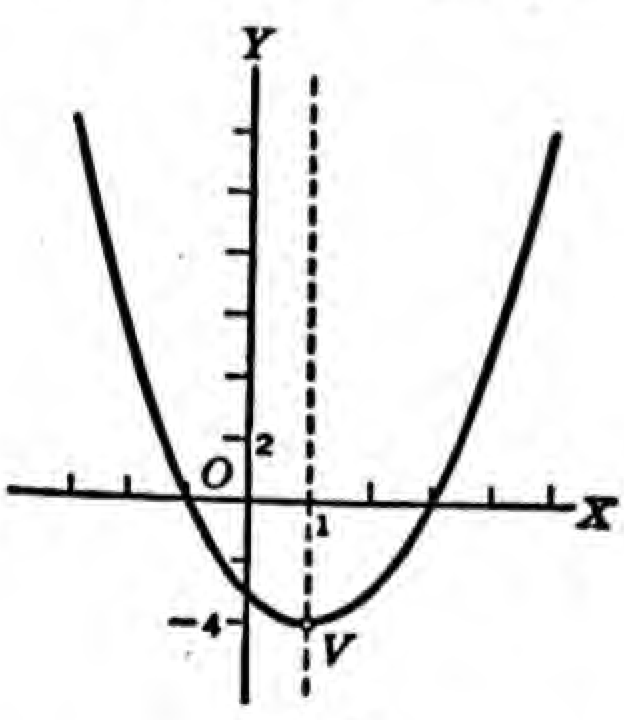


FIG. 7

curve through these points is the graph of the function and is called a **parabola**. The point V at the rounded end is called the **vertex** of the parabola. Since V is the *lowest* point of the graph, the ordinate of V, or -4, is the *smallest* or **minimum value of the function,** and we call V the **minimum point** of the graph. The vertical line through V is called the **axis** of the parabola. The part of the curve to the right of this axis has exactly the same shape as the part to the left. That is, the parabola is *symmetrical* with respect to its axis. The equation of the axis of the parabola $y = x^2 - 2x - 3$ shown in Figure 7 is $x = 1$.

If a parabola is concave downward, instead of upward as shown in Figure 7, then the vertex of the parabola is its *highest* point and would be called the **maximum point** of the curve.

Note 1. A parabola can be defined geometrically as the curve of intersection when a right circular cone is cut by a plane which is parallel to a straight line on the cone through its apex.

At a more advanced stage, we meet proofs of the following facts:

I. *The graph of* $\mathbf{ax^2 + bx + c}$ *is a parabola with its axis perpendicular to the x-axis; this parabola is* **concave upward if a is positive** *and* **concave downward if a is negative.**

II. *The abscissa of the vertex of the parabola is* $\mathbf{x} = -\dfrac{b}{2a}$; *when x has this value, the function has its* **minimum** *or its* **maximum** *value according as* $\mathbf{a}$ *is* **positive** *or* **negative.**

ILLUSTRATION 1. In Figure 7, at V, $x = -\dfrac{-2}{2\cdot 1} = 1.$

To form a table of values in graphing a quadratic function $f(x)$:

1. *Find the coordinates of the vertex of the graph.*

2. *Choose pairs of values of x where, in each pair, the values are equidistant from the vertex, one value on each side; the values of $f(x)$ corresponding to each pair will be equal.*

79. Graphical solution of an equation. If x has a value for which the graph of $f(x)$ meets the x-axis, then with this value of x we have $f(x) = 0$. Hence we are led to the following procedure for finding approximate values of the real roots of an equation in x graphically:

1. *Simplify and transpose all terms to one member to obtain an equation of the form $f(x) = 0$.*

2. *Graph $f(x)$ and measure the abscissas of the points where the graph meets the x-axis; each of these abscissas satisfies $f(x) = 0$.*

EXAMPLE 1. Solve $x^2 - 2x - 3 = 0$ graphically.

SOLUTION. 1. Let $y = x^2 - 2x - 3$ and consider its graph in Figure 7, page 71. The graph crosses the x-axis at $x = 3$ and $x = -1$.

2. Since $y = 0$ when $x = 3$ and when $x = -1$, these are values of x for which $x^2 - 2x - 3 = 0$. That is, 3 and -1 are roots of the equation.

80. The graphical solution of a quadratic equation

$$ax^2 + bx + c = 0 \tag{1}$$

is obtained by use of the graph of the quadratic function

$$ax^2 + bx + c. \tag{2}$$

The parabola, which is the graph of this function,

I. *cuts the x-axis in two points when and only when equation 1 has unequal real roots;*

II. *touches the x-axis in just one point, or is tangent to the x-axis, when and only when the roots are equal;*

III. *does not meet the x-axis when and only when the roots are imaginary.*

ILLUSTRATION 1. In Figure 8, parabolas I, II, and III are, respectively, the graphs of the functions in the left members of the following equations.

$$\text{(I)}\quad x^2 - 2x - 8 = 0;$$
$$\text{(II)}\quad x^2 - 2x + 1 = 0;$$
$$\text{(III)}\quad x^2 - 2x + 5 = 0.$$

We see that (I) has the roots $x = 4$ and $x = -2$; (II) has equal roots, $x = 1$; (III) has imaginary roots.

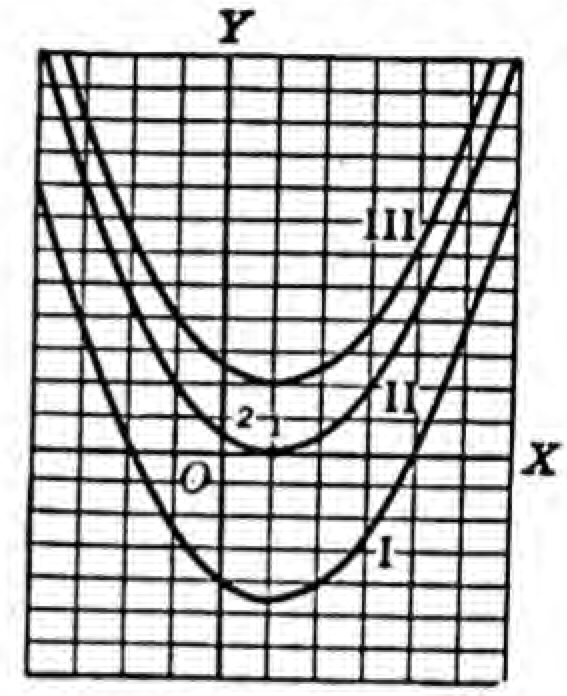


FIG. 8

In graphing a specified quadratic *function*, we have no license to simplify its form by multiplication or by division. But, before solving a quadratic *equation* graphically, we may (1) clear the equation of fractions; (2) divide out any common constant factor from all terms; (3) make the coefficient of x^2 positive. Operation (3) would cause the corresponding graph to open upward.

EXERCISE 27

For each function, (a) find the coordinates of the vertex of the graph and the equation of its axis; (b) graph the function; (c) state the maximum or minimum value of each function.

1. x^2. **3.** $x^2 - 4x + 7$. **5.** $-3x^2 - 6x + 5$. **7.** $4x^2 + 5$.

2. $-x^2$. **4.** $x^2 + 6x + 5$. **6.** $-2x^2 + 8x + 3$. **8.** $8x - 2x^2$.

Find the real roots of the equation graphically.

9. $x^2 + 2x - 8 = 0$. **11.** $x^2 + 4x + 7 = 0$. **13.** $5w = 13 - 2w^2$.

10. $x^2 + 6x + 9 = 0$. **12.** $\frac{5}{2}y = 2 - y^2$. **14.** $2x = -\frac{10}{3} - \frac{1}{3}x^2$.

State whether the function has a maximum or a minimum value, and obtain this value without graphing by finding the coordinates of the vertex of the graph

15. $4x^2 - 8x + 3$. **16.** $-3x^2 + 12x - 2$. **17.** $-6x^2 + 17$.

Graph the equation, with the x-axis **horizontal.** *Give the coordinates of the vertex and the equation of the axis of each parabola.*

18. $y = 4x^2$. **19.** $x = -4y^2 + 2$. **20.** $x = 2y^2 + 8y - 6$.

21. Rewrite statements I and II of page 72 for the case of the graph of $x = ay^2 + by + c$, with the x-axis horizontal in the coordinate system.

22. If an object is shot vertically upward from the earth's surface with an initial velocity of 96 feet per second, (*a*) draw a graph of the distance s as a function of t; (*b*) from the graph, find when the object commences to fall, the maximum height which it reaches, and when it hits the surface. (Recall the formula of Problem 58, page 71.)

★*Solve each problem by introducing just one unknown x and then finding the maximum of a quadratic function of x, without graphing.*

23. Divide 60 into two parts whose product is a maximum.

HINT. Find the maximum of $x(60 - x)$.

24. Find the dimensions of the rectangular field of largest area which can be inclosed with 600 feet of wire fence.

25. In forming a trough with a rectangular cross section and open top, a sheet of tin is bent upward on each long side. If the sheet is 30 inches wide, find the dimensions of the cross section with the largest area.

81. Character of the roots. Let r and s represent the roots of $ax^2 + bx + c = 0$. Then, from the quadratic formula,

$$r = \frac{-b + \sqrt{b^2 - 4ac}}{2a}; \qquad s = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{1}$$

We assume that a, b, and c are real numbers and that $a \neq 0$. Then, the roots are *imaginary* when and only when $b^2 - 4ac$ is *negative;* if one root is imaginary, the other is also.

If $b^2 - 4ac = 0$, then $r = s = -b/2a$. Moreover, if $r = s$, on subtracting the expressions in (1) we obtain

$$0 = r - s = \frac{2\sqrt{b^2 - 4ac}}{2a}; \qquad \frac{\sqrt{b^2 - 4ac}}{a} = 0; \qquad \sqrt{b^2 - 4ac} = 0.$$

Hence, if $r = s$ then $b^2 - 4ac = 0$.

From the preceding remarks and Section 80, we see that the items in any row of the following summary hold simultaneously.

THE ROOTS OF $ax^2 + bx + c = 0$	THE VALUE OF $b^2 - 4ac$	THE GRAPH OF $ax^2 + bx + c$
real and unequal	$b^2 - 4ac > 0$	*cuts x-axis in two points*
real and equal	$b^2 - 4ac = 0$	*is tangent to x-axis*
imaginary	$b^2 - 4ac < 0$	*does not touch x-axis*

If a, b, and c are *rational* numbers, the roots are rational when and only when $\sqrt{b^2 - 4ac}$ is *real* and is a rational number. That is, *the roots are* **rational** *when and only when* $b^2 - 4ac$ *is a* **perfect square.**

We call $b^2 - 4ac$ the **discriminant** of the *quadratic equation* $ax^2 + bx + c = 0$, or of the *quadratic function* $ax^2 + bx + c$, because, as soon as we know the value of $b^2 - 4ac$, we can tell the general character of the roots of the equation without solving it, and the general nature of the graph of the function without graphing it.

ILLUSTRATIONS OF THE USE OF THE DISCRIMINANT

EQUATION	DISCRIMINANT	HENCE, THE ROOTS ARE
$4x^2 - 3x + 5 = 0$	$(-3)^2 - 4 \cdot 4 \cdot 5 = -71$	*imaginary numbers*
$4x^2 - 4x + 1 = 0$	$4^2 - 4 \cdot 4 = 0$	*real; equal; rational*
$4x^2 - 3x - 5 = 0$	$(-3)^2 + 4 \cdot 4 \cdot 5 = 89$	*real; unequal; irrational*
$x^2 - 2x - 3 = 0$	$(-2)^2 - 4(-3) = 16 = 4^2$	*real; unequal; rational*

EXAMPLE 1. Find the values of k for which the following equation in x has equal roots: $kx^2 + 2x^2 - 3kx + k = 0$.

SOLUTION. 1. In standard form: $(k + 2)x^2 - 3kx + k = 0$. Hence, the standard coefficients are $a = k + 2$, $b = -3k$, and $c = k$.

2. If the roots are equal, the discriminant $b^2 - 4ac$ is zero:

$$\textit{discriminant} = (-3k)^2 - 4(k + 2)(k) = 0; \quad \text{or} \quad 5k^2 - 8k = 0.$$

3. Hence, $k(5k - 8) = 0$; or $k = 0$ and $k = \frac{8}{5}$.

EXAMPLE 2. State what you can learn about the graph of the quadratic function $-3x^2 + 5x - 6$ *without* graphing.

SOLUTION. The discriminant of the function is $25 - 72 = -47$. Hence, the graph would *not* touch the x-axis. Since the coefficient of x^2 is -3, the graph is concave *downward* and therefore lies wholly *below* the x-axis.

82. Conjugate imaginaries. If two imaginary numbers differ only in the *signs of the coefficients of their imaginary parts,* then either of the given numbers is called the *conjugate* of the other.

ILLUSTRATION 1. The conjugate of $(3 + 5i)$ is $(3 - 5i)$.

When the roots of a quadratic equation are imaginary, *these roots are conjugate imaginary numbers,* because the imaginary parts come from $\pm \sqrt{b^2 - 4ac}$ in the quadratic formula.

ILLUSTRATION 2. The roots of $x^2 + 4x + 5 = 0$ are $x = -2 \pm i$.

EXERCISE 28

Compute the discriminant and tell the character of the roots, without solving.

1. $y^2 - 7y + 10 = 0$.
2. $x^2 + 2x - 2 = 0$.
3. $3x^2 - 5x + 7 = 0$.
4. $9x^2 + 12x + 4 = 0$.
5. $4x^2 + 4x = 3$.
6. $5x^2 + 1 = 2x$.
7. $25x^2 + 1 = -10x$.
8. $1 = 2x - 2x^2$.
9. $3 + 5x^2 = 0$.

Solve graphically; check the graph by computing the discriminant.

10. $x^2 - 4x = 6$.
11. $x^2 + 7 = 4x$.
12. $4x^2 + 4x = 1$.

Compute the discriminant of the function and, without graphing, state all facts which you can learn about its graph.

13. $4x^2 - 12x + 9$.
14. $2x^2 - 3x - 5$.
15. $3x^2 - 4x$.
16. $-3x^2 + 5x - 7$.
17. $4x^2 + 5x + 7$.
18. $-3x^2 - 2x + 4$.

By use of the discriminant, find the values of the constant k for which the equation will have equal roots for the unknown x.

19. $4x^2 - 3kx + 1 = 0$.
20. $kx^2 + 3kx + 5 = 0$.
21. $x^2 - kx^2 - 5kx - 3k = 0$.
22. $kx + x^2 + kx^2 - 2x = 4$.

Find the values of the constant k for which the graph of the function of x will be tangent to the x-axis.

23. $5x^2 - 2kx + k$.
24. $x^2 - 3x - k - kx$.

★25. Prove that, if a and c are *opposite* in sign, the roots of the equation $ax^2 + bx + c = 0$ are real and unequal.

83. The sum and the product of the roots. By use of

$$r = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad s = \frac{-b - \sqrt{b^2 - 4ac}}{2a},$$

we obtain

$$r + s = \frac{-2b}{2a} = -\frac{b}{a};$$

$$rs = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \cdot \frac{-b - \sqrt{b^2 - 4ac}}{2a};$$

$$rs = \frac{(-b)^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}.$$

Hence, for the equation $ax^2 + bx + c = 0$,

the sum of the roots equals $-\frac{b}{a}$: $\qquad r + s = -\frac{b}{a}$. (1)

the product of the roots equals $\frac{c}{a}$: $\qquad rs = \frac{c}{a}$. (2)

ILLUSTRATION 1. For $3x^2 - 5x + 7 = 0$, we find $r + s = \frac{5}{3}$ and $rs = \frac{7}{3}$.

84. Factored form of a quadratic function. *If r and s are the roots of $ax^2 + bx + c = 0$, then*

$$ax^2 + bx + c = a(x - r)(x - s). \tag{1}$$

Proof. 1. We can write $\quad ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right).$

2. From Section 83, $\frac{b}{a} = -(r + s)$ and $\frac{c}{a} = rs$. Hence,

$$ax^2 + bx + c = a[x^2 - (r + s)x + rs] = a(x - r)(x - s).$$

ILLUSTRATION 1. A quadratic equation whose roots are 5 and -3 is

$$(x + 3)(x - 5) = 0, \quad \text{or} \quad x^2 - 2x - 15 = 0. \qquad [a = 1 \text{ in } (1)]$$

ILLUSTRATION 2. A quadratic equation whose roots are $\frac{1}{2}(2 \pm 3i)$ is

$$a[x - \tfrac{1}{2}(2 + 3i)][x - \tfrac{1}{2}(2 - 3i)] = 0.$$

To eliminate fractions we use $a = 4 = 2 \cdot 2$:

$$2\left(x - \frac{2 + 3i}{2}\right) \cdot 2\left(x - \frac{2 - 3i}{2}\right) = (2x - 2 - 3i)(2x - 2 + 3i) = 0;$$

$$[(2x - 2) - 3i][(2x - 2) + 3i] = 0, \quad \text{or} \quad (2x - 2)^2 - 9i^2 = 0;$$

$$4x^2 - 8x + 4 + 9 = 0, \quad \text{or} \quad 4x^2 - 8x + 13 = 0.$$

EXAMPLE 1. Factor $6x^2 - 23x + 20$ by first solving an equation.

SOLUTION. 1. Solve $6x^2 - 23x + 20 = 0$, by the quadratic formula:

$$x = \frac{23 \pm \sqrt{49}}{12} = \frac{23 \pm 7}{12}; \quad x = \frac{5}{2} \quad \text{and} \quad x = \frac{4}{3}.$$

2. From (1), $6x^2 - 23x + 20 = 6(x - \frac{5}{2})(x - \frac{4}{3}) = (2x - 5)(3x - 4)$.

Formula 1 states that *any quadratic function of x can be expressed as a product of factors which are linear in x.* However, these factors involve rational, irrational, or imaginary coefficients depending on the nature of the roots r and s. In particular, from the facts about *rational* roots, on page 75, we draw the following conclusion:

If a, b, and c are rational numbers, $ax^2 + bx + c$ can be expressed as a product of real linear factors with rational coefficients when and only when the discriminant $b^2 - 4ac$ is a perfect square.

85. Miscellaneous problems relating to the roots.

EXAMPLE 1. If c is a constant and 2 is one root of the equation

$$3x^2 - 7x + c = 0,$$

find the other root.

SOLUTION. Let the roots be r and s, with $r = 2$. Then,

$$r + s = \tfrac{7}{3}; \quad \text{or,} \quad 2 + s = \tfrac{7}{3}; \quad s = \tfrac{1}{3}.$$

EXAMPLE 2. Find the constant h if one root of the following equation exceeds the other by 5: $x^2 - x - 2h = 0$.

SOLUTION. 1. Given condition: $r - s = 5$. (1)

2. Sum of the roots: $r + s = 1$. (2)

3. Product of the roots: $rs = -2h$. (3)

4. Solve (1) and (2) for r and s: $r = 3; \quad s = -2$. (4)

5. Substitute (4) in (3): $-6 = -2h; \ h = 3$. (5)

EXERCISE 29

Find the sum and the product of the roots of each equation in the variable x, without solving the equation for x.

1. $x^2 + 5x - 3 = 0$. **3.** $7 - 3x = 4x^2$. **5.** $5 - 9x^2 = 7x$.

2. $-2x^2 + 7 - 5x = 0$. **4.** $5x^2 - 17 = 0$. **6.** $ax^2 + dx = h$.

Form a quadratic equation with integral coefficients having the specified numbers as roots.

7. $3; -7$. **8.** $\frac{1}{3}; 2$. **9.** $-\frac{5}{7}; -\frac{3}{4}$. **10.** $\pm 3\sqrt{3}$. **11.** $\pm 3i$.

12. $1 \pm \sqrt{2}$. **14.** $-6 \pm 3i$. **16.** $\frac{3}{2} \pm \frac{1}{2}\sqrt{-18}$.

13. $-2 \pm 3\sqrt{5}$. **15.** $1 \pm \frac{3}{2}\sqrt{2}$. **17.** $a \pm bi$.

Without factoring or solving any equation, determine whether or not the expression has real linear factors with rational coefficients.

18. $8x^2 + 7x - 2$. **19.** $11x^2 + 12x - 5$. **20.** $6x^2 + 25xy + 25y^2$.

Find the value of the constant h under the given condition.

21. One root is 2: $3hx^2 - 5x + 2h = 0$.

22. One root is 0: $6hx^2 - 3x + 5h^2 - 3h = 0$.

In all problems, x is the unknown and all other letters are constants.

23. If one root is 3, find the other root: $2x^2 - 5x + d = 0$.

24. If one root is -5, find the other root: $2x^2 + bx - 3 = 0$.

Find the value of the constant h under the given condition.

25. The sum of the roots is -5: $3hx^2 - 4x - 5hx + 6 = 0$.

26. The product of the roots is 9: $2x^2 - 3hx^2 - 6x + 4h = 0$.

27. The product of the roots is 6: $3hx^2 + 5x + h = 1$.

28. One root exceeds the other by 2: $2x^2 - 4h + 5x = 0$.

29. One root is four times the other: $2x^2 + 20x + h^2 = 13$.

30. One root is three times the other: $2x^2 - 16x + 3h = 1$.

Factor by first solving a quadratic equation for x by use of the quadratic formula.

31. $12x^2 + 11x - 36$. **33.** $27x^2 - 6xy - 16y^2$. **35.** $4x^2 - 12x + 7$.
32. $27x^2 + 21x - 40$. **34.** $x^2 + 6x + 10$. **36.** $4x^2 - 4x + 10$.

★*Prove the following theorems about roots of $ax^2 + bx + c = 0$.*

37. If one root is the negative of the other, then $b = 0$.

38. If $b = 0$, then one root is the negative of the other.

Note 1. Problem 38 is the *converse* of Problem 37. If the words *and conversely* had been added to Problem 37, this would have required us to prove both theorems as now stated in the two problems.

39. If $b = 0$ and $c = 0$, then both roots are zero, *and conversely.*

40. If a and c are of opposite sign, then one root is positive and one is negative, *and conversely.*

★*Solve the following problems by use of one of the preceding theorems.*

Find the constant h if one root is the negative of the other.

41. $hx^2 - 6x + 3hx - 5 = 0$. **42.** $2hx^2 - 3hx - 5h^2x + 6 = 0$.

Find the constants h and k if both roots of the equation are zero.

43. $5x^2 + 2hx - 3kx + h - x - 3k - 1 = 0$.

86. Equations in quadratic form.

EXAMPLE 1. Solve: $$x^4 - 5x^2 + 6 = 0.$$

SOLUTION. 1. Factor: $$(x^2 - 3)(x^2 - 2) = 0.$$

2. If $x^2 - 3 = 0$, then $x = \pm\sqrt{3}$; if $x^2 - 2 = 0$, then $x = \pm\sqrt{2}$. The given equation has four solutions, $\pm\sqrt{3}$ and $\pm\sqrt{2}$.

Comment. The given equation is said to be in the *quadratic form* in x^2 because we would obtain a quadratic in y on substituting $y = x^2$.

EXAMPLE 2. Solve: $2x^{-4} - x^{-2} - 3 = 0$.

SOLUTION. 1. Let $y = x^{-2}$; then $y^2 = x^{-4}$ and $2y^2 - y - 3 = 0$.

2. Solve for y: $(2y - 3)(y + 1) = 0$; hence, $y = -1$ and $y = \frac{3}{2}$

3. If $y = \dfrac{3}{2}$, then $x^{-2} = \dfrac{3}{2}$; $\dfrac{1}{x^2} = \dfrac{3}{2}$; $x^2 = \dfrac{2}{3}$; $x = \pm\dfrac{1}{3}\sqrt{6}$.

4. If $y = -1$, then $x^{-2} = -1$; hence, $x^2 = -1$; $x = \pm i$.

EXAMPLE 3. Solve: $(x^2 + 3x)^2 - 3x^2 - 9x - 4 = 0$.

INCOMPLETE SOLUTION. 1. Group: $(x^2 + 3x)^2 - 3(x^2 + 3x) - 4 = 0$.

2. Let $y = x^2 + 3x$; then $y^2 - 3y - 4 = 0$; hence, $y = 4$, and $y = -1$. We then would solve $x^2 + 3x = 4$ and $x^2 + 3x = -1$.

Note 1. In solving an equation of the form $x^k = A$ where k is a positive integer greater than 2, we agree for the present that we desire only *real* solutions *unless otherwise specified.* The real solutions, if any, of $x^k = A$ are the real kth roots of A. Thus, $x^4 = -8$ has no real solutions while $x^6 = 64$ has the real solutions $x = \pm\sqrt[6]{64} = \pm 2$.

EXAMPLE 4. Obtain all roots by use of factoring: $8x^3 + 125 = 0$.

SOLUTION. 1. Factor: $(2x + 5)(4x^2 - 10x + 25) = 0$.

2. Hence, $\quad 2x + 5 = 0, \quad$ or $\quad 4x^2 - 10x + 25 = 0$.

3. $\quad x = -\frac{5}{2}$ and $x = \frac{1}{8}(10 \pm \sqrt{100 - 400}) = \frac{5}{4} \pm \frac{5}{4}i\sqrt{3}$.

EXAMPLE 5. Find the four 4th roots of 625.

INCOMPLETE SOLUTION. If x is any 4th root of 625, then $x^4 = 625$.

In this section the student has met further illustrations of the truth of the fundamental theorem of algebra that *an integral rational equation of degree n in a single variable x has exactly n roots* (we admit the possibility that some of the roots may be equal).

EXERCISE 30

Solve by the method of Section 86 *without first clearing of fractions when they occur. Results may be left in simplest radical form.*

1. $x^4 - 13x^2 + 36 = 0$. **4.** $3x^4 + 11x^2 = 4$. **7.** $16y^4 - 81 = 0$.

2. $x^4 - 26x^2 + 25 = 0$. **5.** $16x^4 = 8x^2 - 1$. **8.** $4y^{-4} - 5y^{-2} + 1 = 0$.

3. $2x^4 + 17x^2 - 9 = 0$. **6.** $x^6 + 7x^3 = 8$. **9.** $9x^{-4} - 13x^{-2} + 4 = 0$.

10. $y^{-4} - 3y^{-2} + 2 = 0$. **13.** $(x^2 + x)^2 - 8(x^2 + x) + 12 = 0$.

11. $2x^{-4} - 11x^{-2} + 5 = 0$. **14.** $(x^2 - 3x)^2 - (3x^2 - 9x) = 4$.

12. $8x^6 - 35x^3 + 27 = 0$. **15.** $3(x^2 + 2x)^2 - 8x^2 - 16x - 3 = 0$.

16. $\dfrac{4}{(x+1)^2} + \dfrac{7}{x+1} - 2 = 0$. **17.** $\left(x - \dfrac{2}{x}\right)^2 + 4\left(x - \dfrac{2}{x}\right) = 5$.

★*Find all roots by first using factoring.*

18. $27x^3 - 8 = 0$. **20.** $x^3 + 8 = 0$. **22.** $16 = 81x^4$.

19. $81 - 625x^4 = 0$. **21.** $8y^3 - 125 = 0$. **23.** $125x^3 + 27 = 0$.

★**24.** Find the three cube roots of (*a*) -27; (*b*) 64; (*c*) 1.

★**25.** Find the four fourth roots of (*a*) 16; (*b*) 81; (*c*) $\frac{1}{625}$.

87. An irrational equation is one where the variables occur under radicals or in expressions with fractional exponents. To solve such an equation, transpose the most complicated radical to one member and all other terms to the other side. Then, proceed as follows.

1. *If the most complicated radical is a square root, square both members; if a cube root, cube both members; etc.*

2. *Repeat Step 1 with the effort to eliminate all radicals involving the unknowns. Then, solve the resulting equation.*

3. *Test each value obtained in Step 2 by substitution in the given equation to determine which values are roots.*

Note 1. Recall that, if A is positive, $\sqrt{A}$, or $A^{\frac{1}{2}}$, represents the *positive* square root of A and that $\sqrt[n]{A}$ represents only the *principal* nth root of A.

EXAMPLE 1. Solve for x in the following equations (a) and (b).

(a) $2x - 2 = \sqrt{2x^2 + 4}$.	(b) $2x - 2 = -\sqrt{2x^2 + 4}$.
SOLUTION. 1. Square both sides:	SOLUTION. 1. Square both sides:
2. $4x^2 - 8x + 4 = 2x^2 + 4$.	2. $4x^2 - 8x + 4 = 2x^2 + 4$.
3. $2x^2 - 8x = 0$; $2x(x - 4) = 0$.	3. $2x^2 - 8x = 0$; $2x(x - 4) = 0$.
4. $x = 0$ and $x = 4$.	4. $x = 0$ and $x = 4$.
TEST. Substitute $x = 0$ in (a):	TEST. Substitute $x = 0$ in (b):
Does $0 - 2 = \sqrt{4}$? Or, does $-2 = 2$? No.	Does $0 - 2 = -\sqrt{4}$? Yes.
Substitute $x = 4$ in (a):	Substitute $x = 4$ in (b):
Does $8 - 2 = \sqrt{36}$? Yes.	Does $8 - 2 = -\sqrt{36}$? Or, does $6 = -6$? No.
$x = 0$ is *not*, and $x = 4$ *is* a root.	$x = 4$ is *not*, and $x = 0$ *is* a root.

If an operation on an equation in x produces a new equation which is satisfied by values of x which are not roots of the given equation, we have agreed to call such values **extraneous roots.** From Example 1, we observe that, if both sides of an equation are squared,* extraneous roots may be introduced.

Note 2. We met the extraneous roots $x = 0$ in solving (a) and $x = 4$ in solving (b). The test of the values obtained in Step 4 in either solution was necessary in order to *reject* these extraneous roots. The necessity for the test is also shown by the fact that, although (a) and (b) are *different equations,* all distinction between them is lost after squaring.

★EXAMPLE 2. Solve: $x^2 + 5 + 2(x^2 + 5)^{\frac{1}{2}} = 15$.

INCOMPLETE SOLUTION. 1. The equation is in the quadratic form in $(x^2 + 5)^{\frac{1}{2}}$. We let $y = (x^2 + 5)^{\frac{1}{2}}$. Then, the original equation becomes

$$y^2 + 2y = 15; \quad y = -5 \quad \text{and} \quad y = 3.$$

2. If $y = 3$, then $\sqrt{x^2 + 5} = 3$; $x = \pm 2$; etc.

* Also true if both sides are raised to any positive integral power.

EXAMPLE 3. Solve: $(x-2)^{\frac{1}{2}} - \sqrt{2x+5} = 3.$

SOLUTION. 1. $\sqrt{x-2} = 3 + \sqrt{2x+5}.$

2. Square: $x - 2 = 9 + 6\sqrt{2x+5} + 2x + 5.$

3. Simplify: $-x - 16 = 6\sqrt{2x+5}.$

4. Square: $x^2 + 32x + 256 = 36(2x+5);$

$$x^2 - 40x + 76 = 0; \quad (x-38)(x-2) = 0.$$

Possible roots of the given equation are $x = 38$ and $x = 2$.

TEST. $x = 2$: does $\sqrt{2-2} - \sqrt{4+5} = 3$, or does $-3 = 3$? No.

$x = 38$: does $\sqrt{38-2} - \sqrt{76+5} = 3$, or does $6 - 9 = 3$? No.

Hence, neither $x = 2$ nor $x = 38$ is a root. Therefore there are *no solutions.*

EXERCISE 31

Solve.

1. $\sqrt{x+3} = 5.$
2. $\sqrt{3-x} = -2.$
3. $\sqrt[4]{1-y} = 2.$
4. $(2+3x)^{\frac{1}{3}} = 2.$
5. $3x = 2\sqrt{5}.$
6. $\sqrt{y} = y - 6.$
7. $3\sqrt{x} = 9 - 2x.$
8. $5\sqrt{y} - 3 = -2y.$
9. $\sqrt[4]{x^2 - 24x} = 3.$
10. $\sqrt{2x+4} = \sqrt{2x} + 1.$
11. $\sqrt{5+2x} = 1 + \sqrt{2x}.$
12. $\sqrt{2x+4} + \sqrt{2x} = 1.$
13. $\sqrt{y-2} - \sqrt{2y+3} = 2.$
14. $\sqrt{7+2x} - \sqrt{3+x} = 1.$
15. $\sqrt{3+2x} + \sqrt{2-2x} = 3.$
16. $\sqrt{2-4x} + 2\sqrt{1-3x} = 2.$
17. $\sqrt{3-x} - (3+x)^{\frac{1}{2}} = \sqrt{x}.$
18. $\sqrt{2x + \sqrt{2x-4}} = 2.$
19. $\sqrt{2\sqrt{x+5} - \sqrt{x}} = 2.$
20. $\sqrt{3+3x} = 2\sqrt{3x-2} + \sqrt{3-x}.$

21. Solve $v = \sqrt{2gs}$, (*a*) for s; (*b*) for g.
22. Solve for x: $\sqrt{x} + \sqrt{3x+4b} = 2\sqrt{2x+b}.$
23. Solve for z: $\sqrt{z-a} + \sqrt{2z+3a} = \sqrt{5a}.$

★24. Solve: $4x^{\frac{2}{3}} + 7x^{\frac{1}{3}} - 2 = 0.$

HINT. 1. Let $y = x^{\frac{1}{3}}$; then $y^2 = x^{\frac{2}{3}}$ and $4y^2 + 7y - 2 = 0.$

★*Find all real roots.*

25. $5z + 3\sqrt{z} = 2.$
26. $3x + 7x^{\frac{1}{2}} = 6.$
27. $2x^{\frac{1}{2}} + 9x^{\frac{1}{4}} = 5.$
28. $2x^{\frac{2}{3}} = 6 + x^{\frac{1}{3}}.$
29. $3x^{-1} + 5 = 8x^{-\frac{1}{2}}.$
30. $2x^{-1} + x^{-\frac{1}{2}} = 6.$
31. $x^2 + 2x - \sqrt{x^2 + 2x - 6} = 12.$
32. $2x^2 + 3\sqrt{2x^2+3} = 7.$
33. $y^{\frac{3}{2}} = 8.$
34. $x^{\frac{3}{4}} = -8.$
35. $(5-3x)^{\frac{3}{4}} = 27.$
36. $(2+3x)^{\frac{3}{2}} = 8.$
37. $2x^{-3} + 15x^{-\frac{3}{2}} - 8 = 0.$
38. $3x^3 + 26x^{\frac{3}{2}} - 9 = 0.$

CHAPTER SIX

Systems Involving Quadratics

88. Graph of a quadratic equation in two variables. A **solution** of an equation in two variables x and y is a **pair of values** of the variables which satisfies the equation. The *graph* or *locus* of the equation is the set of all points whose coordinates, (x, y), form real-valued solutions of the equation.

EXAMPLE 1. Graph: $\qquad x^2 + y^2 = 25.$

SOLUTION. 1. When $x = 0$, $y^2 = 25$; $y = \pm 5$. Two solutions of the equation are $(0, 5)$ and $(0, -5)$. When $y = 0$, $x^2 = 25$; $x = \pm 5$. Two more solutions are $(5, 0)$ and $(-5, 0)$.

2. We plot the four points just found, with the same unit on OX and OY in Figure 9, and verify an advance inference that the graph is a *circle*.

Comment. Let P, with coordinates (x, y), be any point in the coordinate plane with origin at O, in a coordinate system where the *same unit* is used in measuring *all* lengths. Then $x^2 + y^2 = (OP)^2$. Hence, if $x^2 + y^2 = 25$, P lies on a circle with O as center and radius 5.

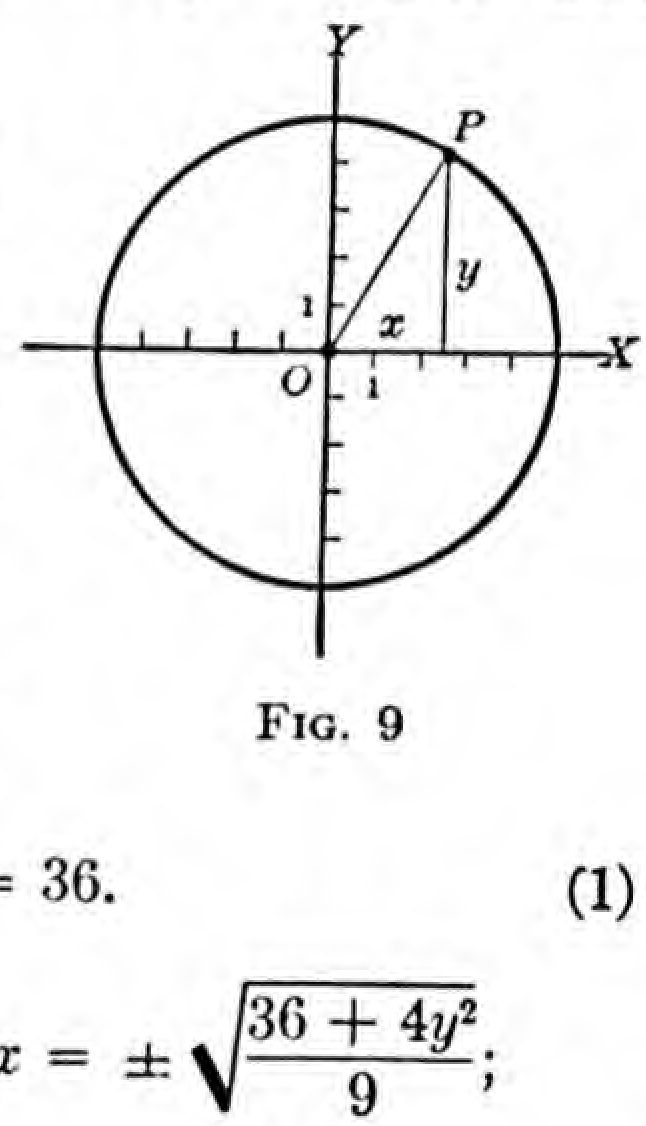


FIG. 9

EXAMPLE 2. Graph:
$$9x^2 - 4y^2 = 36. \tag{1}$$

SOLUTION. 1. Solve for x: $\qquad x^2 = \dfrac{36 + 4y^2}{9};\quad x = \pm\sqrt{\dfrac{36 + 4y^2}{9}};$

$$x = \pm \tfrac{2}{3}\sqrt{9 + y^2}. \tag{2}$$

2. We assign values to y and compute x. Thus, if $y = 0$, then $x = \pm \frac{2}{3}\sqrt{9} = \pm 2$. If $y = 3$, then $x = \pm \frac{2}{3}\sqrt{18} = \pm 2\sqrt{2} = \pm 2.8$.

(*a*)	$y =$	-6	-3	0	3	6
$x = \frac{2}{3}\sqrt{9 + y^2}$	$x =$	4.5	2.8	2	2.8	4.5
(*b*)	$y =$	-6	-3	0	3	6
$x = -\frac{2}{3}\sqrt{9 + y^2}$	$x =$	-4.5	-2.8	-2	-2.8	-4.5

We plot the points given by the pairs of values of x and y in the table. In Figure 10, the points listed for (*a*) in the table give the open curve *FDE*; the points for (*b*) give *HBG*. These two open curves, together, are called a **hyperbola,** and it is the graph of equation 1. Each piece of the hyperbola is called a *branch* of it.

Comment. The equation $9x^2 - 4y^2 = 36$ defines x as a *two*-valued function of y, as shown in (2), or y as a *two*-valued function of x. The graph of the equation consists of the graphs of the two *single*-valued irrational functions

$$x = +\tfrac{2}{3}\sqrt{9 + y^2} \quad\text{and}\quad x = -\tfrac{2}{3}\sqrt{9 + y^2}.$$

The graph of the first of these is the branch *FDE* and the graph of the second is *HBG*. The two branches *together* make up the graph of equation 1. The branches are *symmetrical* with respect to the y-axis and to the x-axis.

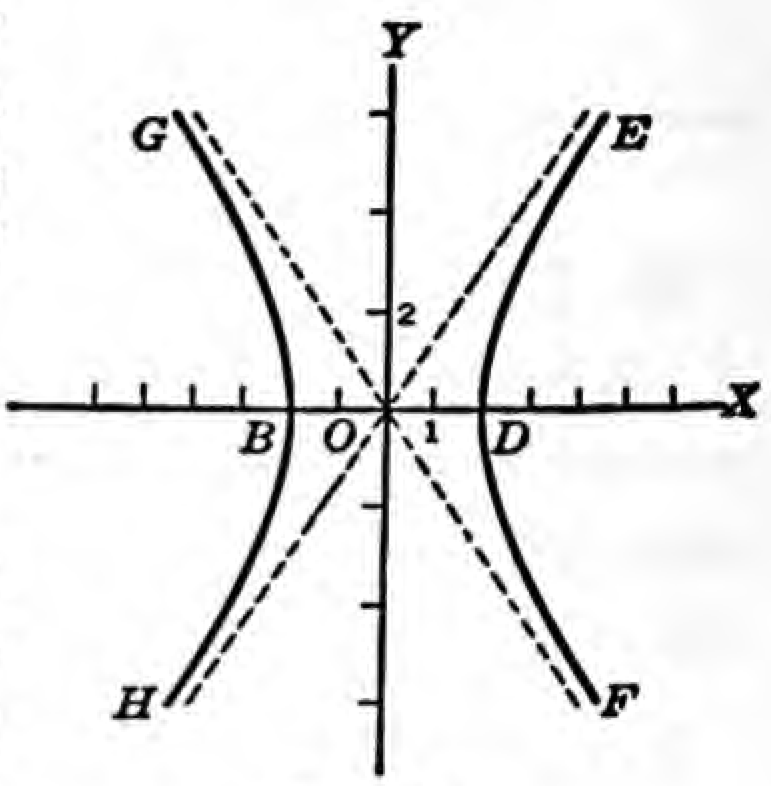


FIG. 10

Note 1. To every hyperbola there correspond two characteristic lines, called **asymptotes,** indicated by dotted lines in Figure 10. As we recede out on any branch of the hyperbola, the curve approaches the corresponding asymptote but *never reaches it.* By moving far enough out on the branch, we may approach the asymptote as closely as we please. It is proved in analytic geometry that the equations of the asymptotes for equation **1** are obtainable as follows:

1. *Replace the constant term in the equation by* 0, *and factor the left member:*

$$9x^2 - 4y^2 = 0; \quad (3x - 2y)(3x + 2y) = 0.$$

2. *Equate each factor separately to zero:*

$$3x - 2y = 0 \quad\textit{and}\quad 3x + 2y = 0.$$

These are the equations of the asymptotes.

EXAMPLE 3. Graph: $x^2 + 4y^2 = 25.$ (3)

SOLUTION. 1. Solve for y: $y^2 = \frac{1}{4}(25 - x^2)$; $y = \pm\frac{1}{2}\sqrt{25 - x^2}$.

2. To obtain real values for y, the numerical value of x may not be allowed to exceed 5. Thus, if $x = 8$, $\sqrt{25 - x^2} = \sqrt{-39}$, which is imaginary.

3. On substituting values for x from $x = -5$ to $x = +5$, and computing y, we obtain solutions which, when plotted and joined by a smooth curve, give the oval *ABCD* in Figure 11, page 86. This curve is called an **ellipse.** The graph of $y = +\frac{1}{2}\sqrt{25 - x^2}$ is the half of this ellipse above the x-axis; the graph of $y = -\frac{1}{2}\sqrt{25 - x^2}$ is the lower half, symmetrical to the upper half. The *whole* ellipse is the graph of equation 3.

EXAMPLE 4. Determine the nature of the graph of

$$4y - 3x^2 + 5x - 7 = 0.$$

SOLUTION. 1. Solve for y: $$y = \tfrac{3}{4}x^2 - \tfrac{5}{4}x + \tfrac{7}{4}.$$

2. Thus, y is a *quadratic function* of x, and therefore the graph of the given equation is a *parabola* whose axis is parallel to the y-axis.

SUMMARY.* *The graph of any quadratic equation in two variables x and y with real solutions is either an ellipse, a hyperbola, a circle, a parabola, a pair of straight lines, or a single point.*

1. *If c is positive, the graph of* $\mathbf{x^2 + y^2 = c}$ *is a circle whose radius is $\sqrt{c}$ and center is the origin, provided that the same unit is used on the scales of the x-axis and y-axis.*

2. *If a, b, and c have the same sign, the graph of* $\mathbf{ax^2 + by^2 = c}$ *is an ellipse, with center at the origin; if $a = b$, the ellipse is a circle, provided that the same unit is used on the scales of the x-axis and y-axis.*

3. *If a and b have opposite signs and if c is not zero, the graph of* $\mathbf{ax^2 + by^2 = c}$ *is a hyperbola.*

4. *If $c \neq 0$, the graph of* $\mathbf{xy = c}$ *is a hyperbola; if $c > 0$, one branch of the hyperbola lies wholly in quadrant* I, *and the other in quadrant* III; *if $c < 0$, the branches are in quadrants* II *and* IV, *respectively. The coordinate axes are the asymptotes of the hyperbola.*

5. *If a quadratic equation in x and y does not involve y^2 or xy, the graph of the equation is a parabola whose axis is parallel to the y-axis; if the equation does not involve x^2 or xy, the graph is a parabola whose axis is parallel to the x-axis.*

EXAMPLE 5. Determine the nature of the graph of

$$2x^2 - xy - 3y^2 = 0. \tag{4}$$

SOLUTION. 1. Factor: $(2x - 3y)(x + y) = 0$. Hence,

$$(a)\quad 2x - 3y = 0, \qquad \text{or} \qquad (b)\quad x + y = 0.$$

2. The graphs of (a) and (b) are straight lines through the origin. Hence, the graph of equation 4 is these *two straight lines*.

89. Routine for graphing. It is important to be able to construct reasonably good graphs *quickly*. Beyond this, it is also essential to have a procedure for improving on such graphs if desired. The following suggestions are of aid in constructing graphs quickly for equations of the second degree in x and y.

* The facts of the summary are discussed in analytic geometry.

When $ax^2 + by^2 = c$ is a circle, find its radius, $\sqrt{c/a}$, and construct the circle with compasses.

When $ax^2 + by^2 = c$ is an ellipse, find the x-intercepts and the y-intercepts. Then, sketch the ellipse through the four intercept points.

When $ax^2 + by^2 = c$ is a hyperbola:

Find its asymptotes by replacing c by 0 and constructing the two straight lines which are the graph of $ax^2 + by^2 = 0$.

Find the x-intercepts or the y-intercepts. (One set of intercepts will be imaginary.) Sketch the hyperbola through the real intercepts, with each branch approaching the asymptotes smoothly.

When a quadratic equation in (x, y) is linear in one variable, solve for it and then graph the parabola by the method of Section 78.

Note 1. To improve on a rapid graph, or whenever doubt arises as to the nature of a graph, *solve the given equation for one variable in terms of the other and compute as many points as needed.*

90. Graphical solution of systems involving quadratics.

EXAMPLE 1. Solve the following system graphically:

$$\begin{cases} x^2 - 2y^2 = 1, & (1) \\ x^2 + 4y^2 = 25. & (2) \end{cases}$$

SOLUTION. 1. We graph each equation, on one coordinate system. The graph of (1) is the hyperbola and the graph of (2) is the ellipse in Figure 11.

2. Any point on the hyperbola has coordinates which satisfy (1), and any point on the ellipse has coordinates which satisfy (2). Hence, both equations are satisfied by the coordinates of A, B, C, and D, which are the points of intersection of the ellipse and the hyperbola:

A: $(x = 3, y = 2)$.
B: $(x = -3, y = 2)$.
C: $(x = -3, y = -2)$.
D: $(x = 3, y = -2)$.

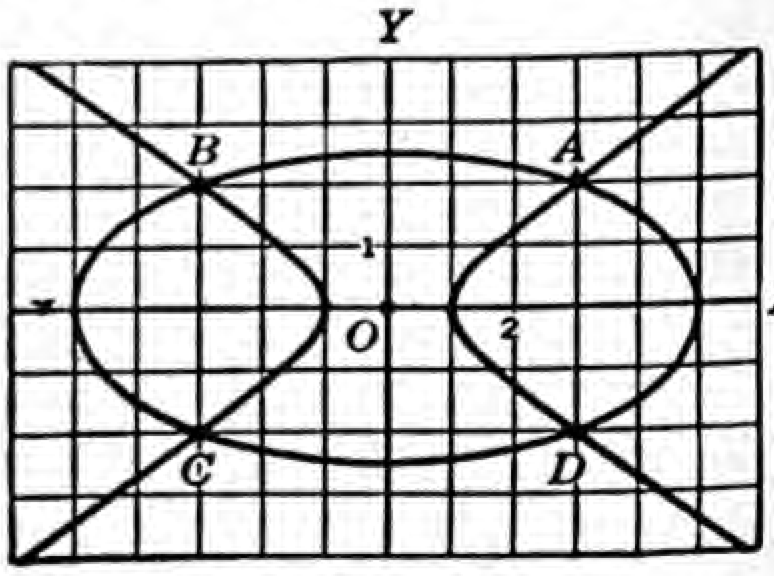


FIG. 11

These pairs of values are the solutions of the system [(1), (2)] and can be checked by substitution in the given equations.

Only real solutions can be found by the preceding graphical method and, usually, solutions can be read only approximately from a graph.

EXERCISE 32

Solve graphically.

1. $\begin{cases} x^2 + y^2 = 16, \\ x - 2y = 3. \end{cases}$

2. $\begin{cases} x^2 + 4y^2 = 4, \\ x + 2y = 1. \end{cases}$

3. $\begin{cases} 4x^2 - y^2 = 16, \\ 3 = x - y. \end{cases}$

4. $\begin{cases} 4x^2 + 9y^2 = 36, \\ 4y^2 - x^2 = 4. \end{cases}$

5. $\begin{cases} 25x^2 + y^2 = 25, \\ 9y^2 - x^2 = 9. \end{cases}$

6. $\begin{cases} x^2 - y^2 = 16, \\ x + y = 2. \end{cases}$

7. $\begin{cases} y = 2x^2 - 8x + 9, \\ xy = 12. \end{cases}$

8. $\begin{cases} 2x^2 - xy - 6y^2 = 0, \\ x^2 + y^2 = 4. \end{cases}$

9. $\begin{cases} x^2 + y^2 = 1, \\ 25x^2 + 4y^2 = 100. \end{cases}$

10. $\begin{cases} 9x^2 + 4y^2 = 36, \\ x^2 + y^2 = 9. \end{cases}$

Graph each equation.

11. $xy = -6$. **12.** $4x^2 + 4y^2 = 9$. **13.** $9x^2 - 16y^2 = 0$. **14.** $x^2 + 4y^2 = 0$.

15. $2x - 4y + 6y^2 = 9$.

16. $(6x - y)(3x + 2y) = 0$.

17. $3x^2 + 4xy - 4y^2 = 0$.

18. $4x^2 + 2y - 6x + 9 = 0$.

91. Algebraic solution of a system of one linear and one quadratic equation in two unknowns. We proceed as follows:

1. *Solve the linear equation for one unknown in terms of the other and substitute the result in the quadratic equation.*

2. *Solve the quadratic equation obtained in Step 1 and, for each value of the unknown obtained, find the corresponding value of the other unknown by substitution in the given linear equation.*

Example 1. Solve:

$$\begin{cases} 4x^2 - 6xy + 9y^2 = 63, & (1) \\ \qquad 2x - 3y = -3. & (2) \end{cases}$$

Solution. 1. Solve (2) for x:

$$x = \frac{3y - 3}{2}. \qquad (3)$$

2. Use (3) in (1):

$$4\left(\frac{3y - 3}{2}\right)^2 - 6y\left(\frac{3y - 3}{2}\right) + 9y^2 = 63. \qquad (4)$$

$$y^2 - y - 6 = 0; \quad (y - 3)(y + 2) = 0; \quad y = 3 \text{ and } y = -2.$$

3. In (3), if $y = 3$, then $x = 3$; if $y = -2$, then $x = -9/2$.

4. The solutions are $\boxed{x = 3, y = 3}$ and $\boxed{x = -\frac{9}{2}, y = -2}$.

Note 1. A system of one linear and one quadratic equation in two unknowns usually has two solutions, either real and distinct, or real and identical, or both imaginary, corresponding respectively, to the following cases: the graph of the linear equation may meet the graph of the quadratic equation in two points, or just one point (be tangent), or in no points. Hereafter, a system of the preceding type will be called a **simple system.**

EXERCISE 33

Solve each system, (a) graphically and (b) algebraically.

1. $\begin{cases} x^2 + y^2 = 25, \\ y + 3x = 15. \end{cases}$ 2. $\begin{cases} a^2 - b^2 = 9, \\ 5a - 4b = 9. \end{cases}$ 3. $\begin{cases} 3a^2 + b^2 = 9, \\ a + b = 4. \end{cases}$

Solve algebraically for x and y.

4. $\begin{cases} 4x^2 - 2xy + y^2 = 7, \\ 2x + y = 4. \end{cases}$
5. $\begin{cases} x + y + 3 = 0, \\ x^2 + 2y^2 - 12y = 18. \end{cases}$
6. $\begin{cases} 5xy = 2x + 2y, \\ 2x + 2y = 5. \end{cases}$
7. $\begin{cases} 4x^2 + 9y^2 = 25, \\ 2x - 7 + 3y = 0. \end{cases}$
8. $\begin{cases} x^2 - 4y^2 = 16, \\ 2y - x = 2. \end{cases}$
9. $\begin{cases} x^2 + 2x - 2y = 23 - y^2, \\ 3x + 4y = 26. \end{cases}$
10. $\begin{cases} xy + 3y + 2x - 1 = 0, \\ x + y + 3 = 0. \end{cases}$
11. $\begin{cases} 4x^2 + y^2 = 2(a^2 + b^2), \\ 2x + y = 2a. \end{cases}$

92. When both equations of a system are quadratic, the system usually has *four different solutions,* all or two of which may involve imaginary numbers. The student should recall his graphical solutions of systems of this type where four solutions were obtained.

Note 1. The fact stated in the preceding paragraph is a special case of the following theorem which is proved in a later course in algebra: *A system of two integral rational equations in x and y, in which one equation is of degree m and the other is of degree n in x and y, usually has mn solutions.* At this stage in algebra, the student is not prepared to consider the solution of *all* systems of simultaneous quadratics. Therefore, in this chapter we consider only special elementary types of systems.

93. When both equations have the form $ax^2 + by^2 = c$, the system is linear in x^2 and y^2 and can be solved for x^2 and y^2 by the methods applicable to systems of linear equations.

EXAMPLE 1. Solve:
$$\begin{cases} x^2 + y^2 = 25, & (1) \\ x^2 + 2y^2 = 34. & (2) \end{cases}$$

SOLUTION. 1. Multiply by 2 in (1): $2x^2 + 2y^2 = 50.$ (3)

2. Subtract, (3) − (2): $x^2 = 16, x = \pm 4.$

3. Substitute $x^2 = 16$ in (1):
$$16 + y^2 = 25; \quad y^2 = 9; \quad y = \pm 3.$$

4. Hence, if x is either $+4$ or -4, we obtain, as corresponding values, $y = +3$ and $y = -3$, and there are *four* solutions of the system.

$\boxed{x = 4, y = 3}$; $\boxed{x = -4, y = 3}$; $\boxed{x = 4, y = -3}$; $\boxed{x = -4, y = -3}$.

EXERCISE 34

Solve each system, (a) graphically and (b) algebraically.

1. $\begin{cases} x^2 + y^2 = 4, \\ x^2 + 9y^2 = 9. \end{cases}$ 2. $\begin{cases} x^2 - 9y^2 = 36, \\ x^2 + y^2 = 36. \end{cases}$ 3. $\begin{cases} 4x^2 - y^2 = 16, \\ 9x^2 + 9y^2 = 16. \end{cases}$

Solve algebraically.

4. $\begin{cases} x^2 + 4y^2 = 14, \\ x^2 = 8y^2 - 16. \end{cases}$ 7. $\begin{cases} 9x^2 = 8y^2 - 6, \\ 8x^2 - 3y^2 = 7. \end{cases}$ 10. $\begin{cases} 7r^2 + 8s^2 = 36, \\ 11r^2 + 5s^2 = -4. \end{cases}$

5. $\begin{cases} x^2 - y^2 = 4, \\ 2x^2 + y^2 = 11. \end{cases}$ 8. $\begin{cases} 15c^2 = 8 + 4d^2, \\ 15 - 12d^2 = 20c^2. \end{cases}$ 11. $\begin{cases} 7x^2 - 6y^2 = 63, \\ 9x^2 + 2y^2 = 13. \end{cases}$

6. $\begin{cases} 2x^2 - 3y^2 = 3, \\ 5x^2 + 2y^2 = 17. \end{cases}$ 9. $\begin{cases} 2t^2 = 6r^2 - 3, \\ 6 = 3t^2 + 5r^2. \end{cases}$ 12. $\begin{cases} 6x^2 + 16 = 9y^2, \\ 4x^2 + 9y^2 = -4. \end{cases}$

94. Reduction to simpler systems by eliminating constants.

EXAMPLE 1. Solve:

$$\begin{cases} x^2 + y^2 = 14, & (1) \\ x^2 - 3xy + 2y^2 = 0. & (2) \end{cases}$$

SOLUTION. 1. Factor (2): $$(x - 2y)(x - y) = 0. \qquad (3)$$

2. Therefore, (2) is satisfied if either $x - 2y = 0$, or $x - y = 0$.

3. Hence, (1) and (2) are satisfied if and only if x and y satisfy one of the following systems; or, we say that the given system [(1), (2)] is equivalent to (I) and (II) below:

$$\text{I. } \begin{cases} x^2 + y^2 = 14, \\ x - y = 0. \end{cases} \qquad \text{II. } \begin{cases} x^2 + y^2 = 14, \\ x - 2y = 0. \end{cases}$$

4. On solving (I) by the method of Section 91, we obtain two solutions: $(x = \sqrt{7}, y = \sqrt{7})$ and $(x = -\sqrt{7}, y = -\sqrt{7})$. From (II) we obtain

$$(x = \tfrac{2}{5}\sqrt{70}, y = \tfrac{1}{5}\sqrt{70}); \qquad (x = -\tfrac{2}{5}\sqrt{70}, y = -\tfrac{1}{5}\sqrt{70}).$$

The preceding method applies if, after writing one given equation with one member zero, we can factor the other member.

EXAMPLE 2. Solve:

$$\begin{cases} x^2 + 3xy = 28, & (4) \\ xy + 4y^2 = 8. & (5) \end{cases}$$

INCOMPLETE SOLUTION. 1. *Eliminate the constants.*

Multiply (4) by 2: $$2x^2 + 6xy = 56. \qquad (6)$$
Multiply (5) by 7: $$7xy + 28y^2 = 56. \qquad (7)$$
Subtract, (6) − (7): $$2x^2 - xy - 28y^2 = 0; \text{ or,}$$
$$(2x + 7y)(x - 4y) = 0. \qquad (8)$$

2. To solve [(4), (5)] we may now solve [(5), (8)]. This system is equivalent to the following simpler systems, which should now be solved:

$$\begin{cases} xy + 4y^2 = 8, \\ 2x + 7y = 0. \end{cases} \qquad \begin{cases} xy + 4y^2 = 8, \\ x - 4y = 0. \end{cases} \qquad (9)$$

Comment. The preceding method, involving *elimination of constant terms and then reduction to simpler systems*, applies sometimes when *all terms in the variables are of the second degree.* Notice that, instead of using (5) in (9), we could equally well have used (4).

Note 1. The substitution method of the following solution of Example 2 and the preceding method are convenient in practically the same cases. The student will probably prefer the method of the first solution.

★SECOND SOLUTION. 1. Let $y = wx$. Then, to find (x, y), we first will find (w, x) to satisfy the following system:

When $y = wx$ in (4): $$x^2 + 3wx^2 = 28. \tag{10}$$

When $y = wx$ in (5): $$wx^2 + 4w^2x^2 = 8. \tag{11}$$

2. Solve each of (10) and (11) for x^2:

$$x^2 = \frac{28}{1 + 3w}. \qquad x^2 = \frac{8}{w + 4w^2}. \tag{12}$$

3. Equate the expressions for x^2: $$\frac{28}{1 + 3w} = \frac{8}{w + 4w^2} \tag{13}$$

4. On solving (13), we find $w = \frac{1}{4}$ and $w = -\frac{2}{7}$.

5. On substituting $w = \frac{1}{4}$ in either equation in (12), we find $x = \pm 4$. Since $y = wx$, we obtain the solutions $(x = 4, y = 1)$; $(x = -4, y = -1)$. By use of $w = -\frac{2}{7}$ we obtain $(x = 14, y = -4)$; $(x = -14, y = 4)$.

EXERCISE 35

Solve algebraically and graphically.

1. $\begin{cases} x^2 + y^2 = 4, \\ (x + y)(x - 2y) = 0. \end{cases}$

2. $\begin{cases} x^2 + y^2 = 9, \\ (x - y)(x + 3y) = 0. \end{cases}$

Solve by reducing to simpler systems unless otherwise directed by the instructor.

3. $\begin{cases} 2x^2 + 5xy - 3y^2 = 0, \\ 2x^2 + 3xy = 2. \end{cases}$

4. $\begin{cases} 2x^2 + 7xy + 6y^2 = 0, \\ x^2 + 3y^2 = 7. \end{cases}$

5. $\begin{cases} x^2 + 3xy = 28, \\ xy + 4y^2 = 8. \end{cases}$

6. $\begin{cases} x^2 - 5xy + 6y^2 = 10, \\ x^2 - xy = 4. \end{cases}$

7. $\begin{cases} 6cd + 2d^2 = -7, \\ 2c^2 - 2cd = 15. \end{cases}$

8. $\begin{cases} x^2 - xy = 1, \\ 2x^2 + 2y^2 = 5. \end{cases}$

9. $\begin{cases} x^2 - xy + 6 = 0, \\ xy + y^2 = 35. \end{cases}$

10. $\begin{cases} m^2 + 2mn = 84, \\ 2mn + n^2 = 64. \end{cases}$

Note 1. Each of the following systems is equivalent to a set of systems of pairs of linear equations.

11. $\begin{cases} (x + y - 1)(x - 2) = 0, \\ 6x^2 + 11xy - 10y^2 = 0. \end{cases}$

12. $\begin{cases} (x + y)(x - y)(x + 3y) = 0, \\ (x + 2y + 1)(x - 2y - 2) = 0. \end{cases}$

13. The sum of the squares of the two digits of a positive integral number is 65 and the number is 9 times the sum of its digits. Find the number.

14. A weight on one side of a lever balances a weight of 6 pounds placed 4 feet from the fulcrum on the other side. If the unknown weight is moved 2 feet nearer the fulcrum, the weight balances 2 pounds placed 9 feet from the fulcrum on the other side. Find the unknown weight.

Solve without first clearing of fractions.

15. $\begin{cases} \dfrac{1}{x} - \dfrac{1}{y} = 11, \\ \dfrac{1}{xy} + 28 = 0. \end{cases}$ **16.** $\begin{cases} \dfrac{1}{x^2} + \dfrac{3}{y^2} = 7, \\ \dfrac{3}{x^2} - \dfrac{2}{y^2} = 10. \end{cases}$ **17.** $\begin{cases} \dfrac{1}{x^2} - \dfrac{3}{y^2} = 1, \\ \dfrac{2}{x^2} - \dfrac{1}{y^2} = 7. \end{cases}$

★95. Additional devices for reducing to simpler systems.

EXAMPLE 1. Solve:
$$\begin{cases} x^3 + y^3 = 27, & (1) \\ x + y = 3. & (2) \end{cases}$$

SOLUTION. 1. Factor (1): $(x + y)(x^2 - xy + y^2) = 27.$ (3)

2. Divide, (3) by (2): $x^2 - xy + y^2 = 9.$ (4)

3. Hence, (x, y) satisfies [(1), (2)] if and only if (x, y) satisfies
$$\begin{cases} x + y = 3, & (5) \\ x^2 - xy + y^2 = 9. & (6) \end{cases}$$

The student should complete the solution by solving [(5), (6)] by the method of Section 91.

EXAMPLE 2. Solve:
$$\begin{cases} x^2 + xy + y^2 = 20, & (7) \\ xy = 5. & (8) \end{cases}$$

INCOMPLETE SOLUTION. 1. Add, (7) + (8): $x^2 + 2xy + y^2 = 25.$ (9)

2. From (9), $(x + y)^2 = 25$; hence $x + y = 5$, or $x + y = -5$.

3. To solve [(7), (8)], we would solve each of the following systems:
$$\begin{cases} x + y = 5, \\ xy = 5. \end{cases} \qquad \begin{cases} x + y = -5, \\ xy = 5. \end{cases}$$

★96. Equations symmetrical in x and y. An equation in x and y is said to be *symmetrical* in x and y in case the equation is unaltered when x and y are *interchanged.* A quadratic equation in x and y is symmetrical in x and y *if the coefficients of x^2 and y^2 are equal and those of x and y are equal.* The method of the next example applies to any system of this type.

EXAMPLE 1. Solve:
$$\begin{cases} x^2 + y^2 + 2x + 2y = 8, & (1) \\ 2xy + x + y = -4. & (2) \end{cases}$$

INCOMPLETE SOLUTION. 1. Substitute $x = u + v; \quad y = u - v.$ (3)

From (1): $$2u^2 + 2v^2 + 4u = 8, \tag{4}$$

From (2): $$2u^2 - 2v^2 + 2u = -4. \tag{5}$$

2. Solve the system [(4), (5)] for u and v:

Eliminate v^2, [(4) + (5)]: $$4u^2 + 6u - 4 = 0. \tag{6}$$

3. Solve (6) for u; then obtain v from (4). Each pair of values (u, v) when placed in (3) gives a solution of [(1), (2)].

★EXERCISE 36

Solve by any convenient method.

1. $\begin{cases} 2a + b = 2, \\ 8a^3 + b^3 = 98. \end{cases}$ 2. $\begin{cases} x^2 + xy + y^2 = 7, \\ x^3 - y^3 = 35. \end{cases}$ 3. $\begin{cases} x(x + y) = 40, \\ y(x + y) = 20. \end{cases}$

4. $\begin{cases} x^2y + 2xy^2 = -24, \\ x + 2y - 4 = 0. \end{cases}$ 7. $\begin{cases} 4x^2 + 3xy + y^2 = 8, \\ xy = 1. \end{cases}$

5. $\begin{cases} x^2 + 2xy + y^2 = 4, \\ xy + 3x + 6 = 0. \end{cases}$ 8. $\begin{cases} (x + y)^2 + x + y = 12, \\ x^2 + y^2 = 5. \end{cases}$

6. $\begin{cases} x^2 + y^2 = 13, \\ xy = 6. \end{cases}$ 9. $\begin{cases} x^2 + xy - 3x = 8, \\ 3y - xy - y^2 = 4. \end{cases}$

10. $\begin{cases} x + 2y + 2z = 3, \\ 2z - 2x - y = 6, \\ x^2 + y^2 + z = 14. \end{cases}$ 11. $\begin{cases} 4x^2 - y^2 - z^2 = 4, \\ 4x^2 - 2y^2 = 1, \\ 4x^2 + y^2 - 5z^2 = 8. \end{cases}$

Find the values of the constant k for which the graphs would be tangent. Then, if k is real, graph the equations of the resulting system.

12. $\begin{cases} x^2 + y^2 = k^2, \\ x + y = 1. \end{cases}$ 13. $\begin{cases} x^2 + 4y^2 = 25, \\ 8y + 3x = k. \end{cases}$ 14. $\begin{cases} x^2 + 4y^2 = 25, \\ 4 + 2y + kx = 0. \end{cases}$

INCOMPLETE SOLUTION of Problem 12. 1. If the graphs of the two equations are tangent, the two solutions of the system must be identical.

2. Substitute $y = 1 - x$ in $x^2 + y^2 = k^2$:

$$2x^2 - 2x + (1 - k^2) = 0. \tag{1}$$

3. From Step 1, we notice that the *discriminant of* (1) *must be zero.*

Find an expression for c in terms of the other constants in case the graphs of the two equations in the variables (x, y) are tangent.

15. $\begin{cases} y = mx + c, \\ 9x^2 + 4y^2 = 36. \end{cases}$ 16. $\begin{cases} y = mx + c, \\ a^2x^2 - b^2y^2 = a^2b^2. \end{cases}$ 17. $\begin{cases} x = my + c, \\ a^2y^2 + b^2x^2 = a^2b^2. \end{cases}$

Solve by the method applying to symmetrical equations.

18. $\begin{cases} x^2 - 4x + y^2 - 4y = 17, \\ xy + 6 = 0. \end{cases}$ 19. $\begin{cases} x^2 - 3xy + y^2 = 1, \\ 2x^2 - xy + 2y^2 = 17. \end{cases}$

MISCELLANEOUS EXERCISE 37

Review of Chapters Five and Six

Graph each of the following functions of x.

1. $3x^2 + 5x - 7$. **2.** $-3x^2 - 4x + 2$. **3.** $-6x^2 - 5$.

Solve, (a) by factoring, (b) graphically, and (c) by the quadratic formula.

4. $3x^2 + x - 4 = 0$. **5.** $8x^2 = 2x + 15$.

Solve for x by the method of completing a square.

6. $2x^2 - 4x = 3$. **7.** $x - 3x^2 = 5$. **8.** $hx^2 + kx = h$.

Without solving, determine (a) the nature of the roots and (b) their sum and their product.

9. $x^2 + 5x = 2$. **10.** $4x - x^2 = 5$. **11.** $4x^2 + 25 = 20x$.

If i represents $\sqrt{-1}$, multiply and simplify.

12. $(3i + 5)(2i - 1)$. **13.** $4i^3(i^2 - 1)$. **14.** $\sqrt{-3}(\sqrt{-5} - 1)$.

Determine the value of the constant h under the given condition.

15. The roots are equal: $9x^2 - 4hx + 3h = 0$.

16. The sum of the roots is 5: $3x^2 - 5x + hx - 2 = 0$.

17. One root is zero: $4x^2 + 7hx - h^2 + 4 = 0$.

Form a quadratic equation having the given numbers as roots.

18. $2, -3$. **19.** $-\frac{3}{4}, \frac{2}{5}$. **20.** $\frac{1}{3}(2 \pm \sqrt{3})$. **21.** $3 \pm 2i$.

Graph each equation.

22. $4x^2 + 9y^2 = 0$. **23.** $4x^2 - 9y^2 = 0$. **24.** $4x^2 - 9y^2 = 36$.

Solve graphically.

25. $\begin{cases} 4x^2 + y^2 = 25, \\ 2x + y = 7. \end{cases}$ **26.** $\begin{cases} x^2 + 4y^2 = 16, \\ x^2 - y^2 = 9. \end{cases}$ **27.** $\begin{cases} 9y^2 - 4x^2 = 36, \\ xy = -4. \end{cases}$

28. $\begin{cases} (x - y - 2)(x - y - 1) = 0, \\ x^2 + y^2 = 10. \end{cases}$ **29.** $\begin{cases} y = 1 - x^2, \\ 4x = y^2 + 2y - 7. \end{cases}$

30–32. Solve Problems 25, 26, and 28 algebraically.

Solve algebraically for x, or for x and y.

33. $\begin{cases} 3x + 2y = -2, \\ xy + 8x = 4. \end{cases}$ **34.** $\begin{cases} 2y^2 - xy = 16, \\ x^2 - xy - y^2 = 20. \end{cases}$

35. $\sqrt{3 - 2x} = 1 - \sqrt{1 - x}$. **37.** $4x^{-4} - 17x^{-2} + 4 = 0$.

36. $9 - 4x^4 - 5x^2 = 0$. **38.** $abx^2 - 2ax + 4 = 2bx$.

39. Find the value of x for which the corresponding value of the function $3x^2 - 6x - 4$ is the least.

CHAPTER SEVEN

Ratio, Proportion, and Variation

97. Ratio. The ratio of a to b is the quotient a/b, which is sometimes written $a{:}b$. A ratio is a fraction, and any fraction can be described as a ratio. The ratio of two concrete quantities has meaning only if they are of the same kind, in which case their ratio is the quotient of their measures in terms of the same unit.

98. Proportion. A *proportion* is a statement that two ratios are equal. That is, a proportion is merely a statement that two fractions are equal. The proportion $a{:}b = c{:}d$ is read "a is to b as c is to d." We say that the four numbers a, b, c, and d *form a proportion.* In a proportion $a{:}b = c{:}d$, the first and fourth numbers, a and d, are called the **extremes,** and the second and third, b and c, are called the **means** of the proportion.

ILLUSTRATION 1. To solve the proportion $x{:}(25 - x) = 3{:}7$, we first change it to fractional form, and then solve the resulting equation:

$$\frac{x}{25 - x} = \frac{3}{7};\quad 7x = 75 - 3x;\quad 10x = 75;\quad \text{hence,}\quad x = 7.5.$$

EXAMPLE 1. Divide 36 into two parts with the ratio 3:7.

SOLUTION. 1. Let x and y be the parts; then $\qquad x + y = 36. \qquad (1)$

2. Also, $x{:}y = 3{:}7$, or $\dfrac{x}{y} = \dfrac{3}{7}$. Hence, $\qquad 7x = 3y. \qquad (2)$

3. On solving the system [(1), (2)] we obtain $(x = 10.8,\ y = 25.2)$.

EXERCISE 38

Express each ratio as a fraction and simplify.

1. $\frac{3}{8}{:}\frac{5}{16}$. 2. $\frac{14}{3}{:}\frac{7}{4}$. 3. $5\frac{1}{2}{:}7\frac{1}{3}$. 4. $x^3y^4{:}x^5y^3$. 5. $az^2{:}a^4z$

Find the ratio of the given quantities.

6. 75 pounds to 160 ounces. 7. 27 days to 156 hours.

Change to fractional form and solve.

8. $3{:}(20 - 2x) = 5{:}2$. 9. $(2 - 3y){:}(4 + 5y) = 3{:}2$.

Solve by introducing one or more unknowns.

10. Divide 45 into two parts whose ratio is $4:11$.

11. Find two numbers whose difference is 18 and whose ratio is $4:3$.

12. The sides of a polygon are 10, 7, 4, and 8 inches long. If the longest side is lengthened by 2 feet, by how much should the other sides be lengthened to obtain a similar polygon?

13. A triangle whose base is 15 inches long has an area of 220 square inches. Find the area of a similar triangle whose base is $6\frac{1}{4}$ feet long.

HINT. The ratio of the areas equals the ratio of the squares of the lengths of a pair of corresponding sides.

14. The area of a quadrilateral is 49 square feet and its longest side is 12 feet long. Find the area of a similar quadrilateral whose longest side is 15 feet long.

15. A man $5\frac{1}{2}$ feet tall stands 40 feet from a street light and casts a shadow $9\frac{1}{2}$ feet long. How high is the light?

If $a:x = x:b$, then x is called a **mean proportional** *between a and b. If $a:x = x:b$, then $x^2 = ab$ or $x = \pm\sqrt{ab}$; if neither a nor b is zero, there are two mean proportionals between a and b. Find the mean proportionals between each of the following pairs of numbers.*

16. 64 and 4.

17. -4 and $-\frac{1}{4}$.

18. 25 and 25.

19. -2 and 8.

20. $2a^3$ and $4a$.

21. y^2 and x^{-4}.

99. Direct variation. Let x and y be related variables. Then, we say that y is *proportional to x*, or that *y varies directly as x*, or that *y is directly proportional to x*, in case there exists a constant k such that, for every value of x, there is a corresponding value of y given by

$$y = kx.$$

We call k the **constant of proportionality** or the **constant of variation.**

ILLUSTRATION 1. The circumference C of a circle varies directly as the radius r because $C = 2\pi r$. The constant of proportionality is 2π.

From $y = kx$, we obtain $k = \dfrac{y}{x}\cdot$ Hence, if y is proportional to x, *the ratio of corresponding values of y and x is a constant.* Thus if y increases by $k\%$, it follows that x also increases by $k\%$.

ILLUSTRATION 2. If y is proportional to x^2, then $y = kx^2$.

If y is any function of x, then y varies when x varies. But, we do not say that y varies *directly* as x except when y is the simple linear function $y = kx$.

100. Inverse variation. We say that y is *inversely proportional to* x, or y *varies inversely as* x, in case there exists a constant k such that, for every value of x, there is a corresponding value of y given by

$$y = \frac{k}{x}.$$

From this equation we obtain $k = xy$, or *the product of corresponding values of* x *and* y *is a constant.*

ILLUSTRATION 1. The time t necessary for a train to go a given distance d varies inversely as the speed s of the train because $t = d/s$.

101. Joint variation. We say that z *varies jointly as* x *and* y, or that z *is directly proportional to* x *and* y, in case z is proportional to the product xy, or

$$z = kxy,$$

where k is a constant of proportionality.

The different types of variation may be combined.

ILLUSTRATION 1. If $P = \dfrac{10x^2y}{z^3}$, then P varies directly as x^2 and y, and inversely as z^3.

EXAMPLE 1. If y is directly proportional to x and w^2, and if $y = 36$ when $x = 2$ and $w = 3$, find y when $x = 3$ and $w = 4$.

SOLUTION. 1. We know that $y = kw^2x$.

2. To obtain k, let $w = 3$, $x = 2$, and $y = 36$: $\quad 36 = 18k;\ \ k = 2.$

3. Hence, $y = 2w^2x$; when $(w = 4, x = 3)$, $\quad y = 2 \cdot 3 \cdot 16 = 96.$

EXERCISE 39

Introduce letters if necessary and express the relation by an equation.

1. H varies directly as x and inversely as w^2.

2. B is proportional to x^2 and inversely proportional to z.

3. Z is proportional to $\sqrt{x}$ and varies inversely as y^2.

4. K is proportional to z and w^2 and inversely proportional to xy.

5. $(x + 2)$ is inversely proportional to $(y + 3)$.

6. The volume of a sphere is proportional to the cube of its radius.

7. The weight of a body above the surface of the earth varies inversely as the square of the distance of the body from the earth's center.

8. The power available in a jet of water varies jointly as the weight of the water per cubic foot, the cube of the water's velocity, and the cross-section area of the jet.

For each formula, give a statement about the variable on the left side in the language of variation. All letters except the constant k represent variables.

9. $y = 7w$. **10.** $z = -3x^2$. **11.** $w = 5xy^2/z$. **12.** $u = 7x^3y/\sqrt{z}$

By employing all data, obtain an equation relating the variables.

13. P is directly proportional to x^2 and $P = 18$ if $x = 4$.

14. R is inversely proportional to x and directly proportional to y, while $R = 4$ when $x = 3$ and $y = 5$.

15. U varies directly as x and y, and inversely as z^2; $U = 15$ when $x = 5$, $y = 2$, and $z = 3$.

16. If w is proportional to x and if $w = 5$ when $x = 7$, find w when $x = -6$.

17. If y is inversely proportional to x and if $y = 5$ when $x = 20$, find y when $x = 15$.

18. If H is proportional to x and inversely proportional to $\sqrt{y}$, and if $H = 3$ when $x = 2$ and $y = 4$, find H when $y = 9$ and $x = 5$.

19. The distance fallen by a body, starting from a position of rest in a vacuum near the earth's surface, is proportional to the square of the number of seconds occupied in falling. If a body falls 256 feet in 4 seconds, how far will it fall in 7 seconds?

20. The kinetic energy E, of a mass of m pounds moving with a velocity v, is proportional to mv^2. If $E = 2500$ foot-pounds when a body weighing 64 pounds is moving at a velocity of 50 feet per second, find the kinetic energy of a body weighing 30 pounds whose velocity is 200 feet per second.

21. If one body is sliding on another, the force of sliding friction is proportional to the normal pressure between the bodies (if this pressure is moderate). If the sliding friction between two cast-iron plates is 60 pounds when the normal pressure is 270 pounds, find the normal pressure when the sliding friction is 600 pounds.

22. The approximate amount of steam per second which will flow through a hole varies jointly as the steam pressure and the area of a cross section of the hole. If 40 pounds of steam per second at a pressure of 200 pounds per square inch flows through a hole whose area is 14 square inches, how much steam at a pressure of 250 pounds per square inch will flow through a hole whose area is 20 square inches?

23. The maximum safe load of a horizontal beam supported at its ends varies directly as its breadth and the square of its depth and inversely as the distance between the supports. If the maximum is 2400 pounds for a beam 4 inches wide and 10 inches deep, with supports 15 feet apart, find the maximum load for a beam of the same material which is 3 inches wide and 5 inches deep, with supports 25 feet apart.

24. How far apart may the supports be placed if a beam 5 inches wide and 8 inches deep, like those in Problem 23, supports 6000 pounds?

25. A beam like those in Problem 23 is 6 inches wide and the supports are 12 feet apart. How deep must the beam be to support 3500 pounds?

26. The electrical resistance of a wire varies as its length and inversely as the square of its diameter. If a wire 350 feet long and 3 millimeters in diameter has a resistance of 1.08 ohms, find the length of a wire of the same material whose resistance is .81 ohm and diameter is 2 millimeters.

27. If y is proportional to x and if $y = 16$ when $x = 4$, graph the relation between x and y.

28. If y is inversely proportional to x and if $y = 16$ when $x = \frac{1}{4}$, graph the relation between x and y.

29. The approximate velocity of a stream of water, necessary to move a round object, is proportional to the product of the square roots of the object's diameter and its specific gravity. If a velocity of 11.34 feet per second is needed to move a stone whose diameter is 1 foot and specific gravity is 4, how large a stone with specific gravity 3 can be moved by a stream whose velocity is 22.68 feet per second?

30. Newton's *Law of Gravitation* states that the force with which each of two masses of m pounds and M pounds attracts the other varies directly as the product of the masses and inversely as the square of the distance between the masses. Find the ratio of the force of attraction when two masses are 8000 miles apart to the force when they are 2000 miles apart.

31. The illumination received from a source of light varies inversely as the square of the distance from the source, and directly as its candle power. At what distance from a 50 candle power light would the illumination be one half that received at 30 feet from a 40 candle power light?

32. The current in an electric circuit varies directly as the electromotive force and inversely as the resistance. In a certain circuit, the electromotive force is A volts, the resistance is b ohms, and the current is c amperes. If the resistance is increased by 20%, what per cent of increase must occur in the voltage to increase the current by 30%?

Note 1. The statement *x is to y is to z as r is to s is to t,* or *x, y, and z are proportional to r, s, and t* is abbreviated by $x:y:z = r:s:t$, and means that there exists a number $k \neq 0$ such that $x = kr$, $y = ks$, and $z = kt$.

★*Find x, y, and z under the given conditions.*

33. $x:y:z = 4:-2:5$, and $x + 2y + z = 40$.

34. $x:y:z = 5:-3:2$, and $x - y - z = 12$.

35. $x:y:z = 3:-1:2$, and $x^2 + y^2 + z^2 = 56$.

36. Divide 2800 into four parts proportional to $5:3:4:2$.

CHAPTER EIGHT

The Binomial Theorem

102. Expansion of $(x + y)^n$. By multiplication, we obtain

$$(x + y)^1 = x + y;$$
$$(x + y)^2 = x^2 + 2xy + y^2;$$
$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3;$$
$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

We see that, if $n = 1$, 2, 3, or 4, *the expansion of* $(x + y)^n$ *contains* $(n + 1)$ **terms** *with the following properties:*

I. *In any term the sum of the exponents of x and y is n.*

II. *The* 1st *term is x^n and the* 2d *term is $nx^{n-1}y$. In each other term the exponent of x is* 1 *less, and of y is* 1 *more, than in the preceding term.*

III. *If the coefficient of any term is multiplied by the exponent of x in that term and is divided by the number of that term, the result is the coefficient of the next term.*

Illustration 1. In $(x + y)^4$, the 3d term is $6x^2y^2$. By use of (III), we obtain $(6 \cdot 2) \div 3$, or 4, as the coefficient of the 4th term.

IV. *The coefficients of terms equidistant from the ends of the expansion are the same.*

The **binomial theorem** states that (I) to (IV) are true if n is any positive integer. This theorem will be proved later.

Example 1. Expand $(c + w)^7$.

Solution. 1. By use of (I) and (II), we obtain

$$(c + w)^7 = c^7 + 7c^6w + \quad c^5w^2 + \quad c^4w^3 + \quad c^3w^4 + \quad c^2w^5 + \quad cw^6 + w^7,$$

where spaces are left for the unknown coefficients.

2. By use of (III), the coefficient of the 3d term is $(7 \cdot 6) \div 2$, or 21; of the 4th term is $(21 \cdot 5) \div 3$, or 35. By use of (IV), we obtain the other coefficients. Hence,

$$(c + w)^7 = c^7 + 7c^6w + 21c^5w^2 + 35c^4w^3 + 35c^3w^4 + 21c^2w^5 + 7cw^6 + w^7.$$

Note 1. The signs are *alternately plus and minus* in the expansion of a power of a binomial where the terms are of *opposite signs.*

ILLUSTRATION 2. By use of (I) to (IV), with $x = 2a$ and $y = -w/3$,

$$\left(2a - \frac{w}{3}\right)^6 = \left[(2a) + \left(-\frac{w}{3}\right)\right]^6$$

$$= (2a)^6 + 6(2a)^5\left(-\frac{w}{3}\right) + 15(2a)^4\left(-\frac{w}{3}\right)^2 + 20(2a)^3\left(-\frac{w}{3}\right)^3$$

$$+ 15(2a)^2\left(-\frac{w}{3}\right)^4 + 6(2a)\left(-\frac{w}{3}\right)^5 + \left(-\frac{w}{3}\right)^6, \quad \text{or}$$

$$\left(2a - \frac{w}{3}\right)^6 = 64a^6 - 64a^5w + \frac{80}{3}a^4w^2 - \frac{160}{27}a^3w^3 + \frac{20}{27}a^2w^4 - \frac{4}{81}aw^5 + \frac{w^6}{729}.$$

Note 2. The following array is called *Pascal's Triangle.* The rows give the coefficients in the successive positive integral powers of $x + y$. To form any row after the second, we first place 1 at the left; the 2d number is the sum of the 1st and 2d numbers in the preceding row; the 3d number in the new row is the sum of the 2d and 3d numbers in the preceding row; etc. This triangle was known to Chinese mathematicians in the fourteenth century. The preceding diagram exhibits the fact that the *largest* coefficient in any power of $x + y$ is the coefficient of the *central term* or *terms.*

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

103. The factorial symbol. The symbol $n!$ is read "*n factorial,*" and is an abbreviation for *the product of all integers from* 1 *to n inclusive,* where n is a positive integer.

ILLUSTRATION 1. $5! = 1\cdot2\cdot3\cdot4\cdot5 = 120.$ $\dfrac{6!}{3!} = \dfrac{1\cdot2\cdot3\cdot4\cdot5\cdot6}{1\cdot2\cdot3} = 120.$

EXERCISE 40

Expand each power by use of Properties I *to* IV.

1. $(x + u)^6$.
2. $(a + b)^5$.
3. $(c - d)^4$.
4. $(w - u)^7$.
5. $(x - 2a)^4$.
6. $(y - 3b)^5$.
7. $(2x - b)^6$.
8. $(3c - d)^4$.
9. $(x + \frac{1}{2}b^2)^4$.
10. $(a^2 - \frac{1}{2}b)^3$.
11. $(x^2 - y^3)^6$.
12. $(x^2 + y^4)^5$.
13. $(x^{\frac{1}{2}} - 2y^{\frac{1}{2}})^5$.
14. $(c + d^{\frac{1}{4}})^4$.
15. $(r^{\frac{1}{3}} + 2)^4$.
16. $(a - x)^8$.
17. $(-x + y^{-1})^4$.
18. $(-y + z^{-2})^3$.
19. $(a^{\frac{1}{3}} - 2x)^5$.
20. $\left(\frac{1}{x} + \frac{1}{y}\right)^4$.
21. $\left(\frac{2}{x} - \frac{1}{b}\right)^3$.
22. $\left(\frac{x}{a} + \frac{a}{b}\right)^6$.
23. $\left(\frac{a}{z} - \frac{z^{\frac{1}{2}}}{y}\right)^5$.

Find the first three terms of the expansion by use of Properties I, II, *and* III.

24. $(x + \frac{1}{2})^{16}$.
25. $(a - 3)^{15}$.
26. $(b^3 + a^3)^{10}$.
27. $(m^2 + 3n)^{20}$.
28. $(1 + \sqrt{2})^{14}$.
29. $(\sqrt{3} - 1)^{12}$.
30. $(x^{\frac{1}{2}} + y^{\frac{1}{2}})^{17}$.
31. $(a^{-1} + x^{-2})^8$.
32. $(a^{-3} - x)^{18}$.
33. $(x + y)^n$.
34. $(x^2 + a)^n$.
35. $(z - w^2)^h$.
36. Compute (*a*) 7!; (*b*) 6!; (*c*) 8! ÷ 4!; (*d*) 5! ÷ 9!.

104. The general term of $(x + y)^n$. By use of Properties (I) to (IV) of page 99,

$$\left.\begin{aligned}(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{1\cdot 2}x^{n-2}y^2 \\ + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3}y^3 + \cdots.\end{aligned}\right\} \tag{1}$$

In (1) we read the dots "$\cdots$" as "*and so forth.*" In observing the terms involving y, y^2, and y^3 in (1), we notice special cases of the following facts, which we shall accept without proof at this stage.

SUMMARY. *Description of the term involving y^r, in $(x + y)^n$:*

A. *The exponent of x is $n - r$.*

B. *The denominator is $r!$.*

C. *The numerator of the coefficient has r factors, the first being n and each other being 1 less than the preceding factor. The last factor is $n - r + 1$.*

When (A), (B), and (C) are combined, they state that*

$$\textit{the term involving } y^r \textit{ is} \quad \frac{n(n-1)\cdots(n-r+1)}{r!}x^{n-r}y^r. \tag{2}$$

By use of formula 2, we may write

$$\left.\begin{aligned}(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \cdots \\ + \frac{n(n-1)\cdots(n-r+1)}{r!}x^{n-r}y^r + \cdots + y^n.\end{aligned}\right\} \tag{3}$$

We refer to (3) as the **binomial formula** and we call (2) the *general term* of the expansion of $(x + y)^n$.

ILLUSTRATION 1. The term involving y^4 in the expansion of $(x + y)^7$ is

$$\frac{7\cdot 6\cdot 5\cdot 4}{4!}x^3y^4 \quad \text{or} \quad 35x^3y^4.$$

* Some teachers may prefer to use that formula for the term in y^r which arises when the theory of combinations is studied later.

EXAMPLE 1. Find the 8th term of $(3a^{\frac{1}{2}} - b)^{11}$.

SOLUTION. The 8th term will involve the 7th power of the 2d term of the binomial. Hence, use (2) with $r = 7$, $x = 3a^{\frac{1}{2}}$, and $y = -b$:

$$\text{8th term is } \frac{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}(3a^{\frac{1}{2}})^4(-b)^7 = -26{,}730a^2b^7.$$

Note 1. To find a formula for the rth term in (3), notice that this term will contain y^{r-1} as a factor. Hence, we substitute $(r-1)$ for r in (2):

$$\textit{the } r\textit{th term is } \frac{n(n-1)\cdots(n-r+2)}{(r-1)!}x^{n-r+1}y^{r-1}. \qquad (4)$$

Example 1 could have been solved by use of (4) with $r = 8$.

EXERCISE 41

Find only the specified term in the expansion.

1. Term involving y^4 in the expansion of $(x + y)^9$.
2. Term involving x^6 in the expansion of $(y^2 - x)^8$.
3. Term involving z^{12} in the expansion of $(x + z^3)^8$.

HINT. Since $z^{12} = (z^3)^4$, use formula 2 with $r = 4$.

4. Term involving y^{10} in the expansion of $(x - y^2)^7$.
5. Term involving y^5 in the expansion of $(x + y)^n$.
6. 4th term of $(a + x)^8$.
7. 3d term of $(w - z)^9$.
8. 5th term of $(x - y)^{12}$.
9. 6th term of $(x^2 + y)^8$.
10. 4th term of $(\frac{1}{2}x - 3y)^8$.
11. 7th term of $(\frac{1}{4}x^{-1} + x)^9$.
12. 5th term of $(x^{-\frac{1}{2}} - 2x)^{13}$.
13. 7th term of $(1 - .1)^8$.
14. Middle term of $(x^2 + y)^8$.
15. Middle term of $(x - y^2)^6$.
16. Middle terms of $(z^{\frac{1}{2}} - x)^7$.
17. Middle terms of $(x^{\frac{1}{2}} - \frac{1}{2}y)^5$.
18. Term involving x^2 in the expansion of $(a - x^{\frac{1}{2}})^6$.
19. Term involving $y^{\frac{4}{3}}$ in the expansion of $(x - y^{\frac{1}{3}})^7$.
20. Term involving $\dfrac{a^4}{x^4}$ in the expansion of $\left(z^2 - \dfrac{2a}{x}\right)^8$.

Find the term or terms with the largest coefficient in each expansion.

21. $(x + y)^6$. 22. $(z - w)^8$. 23. $(a^2 - b)^7$. 24. $(c + d^2)^9$.

Compute by use of the binomial theorem; in problems involving decimals, use only enough terms to get the result accurate to 3 decimal places.

25. $(100 - 1)^4$. 27. 61^3. 29. $(1.02)^{10}$. 31. $.99^4$.
26. 98^4. 28. $(1 + .01)^8$. 30. $(1.03)^9$. 32. $.81^3$.

CHAPTER NINE

Progressions

105. Arithmetic progressions. A *sequence* of things is a set of things arranged in a definite order. An *arithmetic progression* (abbreviated A.P.) is a sequence of numbers called *terms*, each of which, after the first, is derived from the preceding one by adding to it a fixed number called the **common difference.** The common difference can be found by subtracting any term from the one *following* it.

ILLUSTRATION 1. In the arithmetic progression 9, 6, 3, 0, $-3, \cdots$, the common difference is -3. The 6th term would be -6.

106. The *n*th term in an A.P. Let a be the 1st term and d be the common difference. Then, the 2d term is $a + d$; the 3d term is $a + 2d$. In each of these terms, the coefficient of d is 1 *less than the number of the term.* Similarly, the 10th term is $a + 9d$. The nth term is the $(n - 1)$th after the first term, and is obtained after d has been added $(n - 1)$ times, in succession. Hence, if l represents the nth term,

$$l = a + (n - 1)d. \tag{1}$$

ILLUSTRATION 1. If $a = 3$ and $d = 4$, the 18th term is $3 + 17(4) = 71$.

107. The sum of the first *n* terms of an A.P. Let S be the sum. The 1st term is a; the common difference is d; the last term is l; the next to the last term is $l - d$, etc. On writing the sum of the n terms, forward and backward, we obtain

$$S = a + (a + d) + (a + 2d) + \cdots + (l - 2d) + (l - d) + l; \tag{1}$$
$$S = l + (l - d) + (l - 2d) + \cdots + (a + 2d) + (a + d) + a. \tag{2}$$

On adding corresponding sides of (1) and (2) we obtain

$$2S = (a + l) + (a + l) + (a + l) + \cdots + (a + l) + (a + l) + (a + l),$$

where there are n terms $(a + l)$. Hence, $2S = n(a + l)$, or

$$S = \frac{n}{2}(a + l). \tag{3}$$

Example 1. Find the sum of the A.P. $8 + 5 + 2 + \cdots$ to twelve terms

Solution. 1. First obtain l from $l = a + (n - 1)d$. We have

$a = 8$, $d = -3$, and $n = 12$: $\qquad l = 8 + 11(-3) = -25.$

2. From (3), $\qquad S = 6(8 - 25) = -102.$

On substituting $l = a + (n - 1)d$ in (3), we obtain

$$S = \frac{n}{2}[2a + (n - 1)d]. \tag{4}$$

The quantities a, d, l, n, and S are called the **elements** of the general arithmetic progression. When three of the elements are given, we may obtain the other two by use of

$$l = a + (n - 1)d; \quad S = \frac{n}{2}(a + l); \quad S = \frac{n}{2}[2a + (n - 1)d]. \tag{5}$$

Example 2. Find the remaining elements if $a = 2$, $l = 402$, and $n = 26$.

Solution. 1. We wish to find d and S. From (3), $S = 13(404) = 5252$.

2. From $l = a + (n - 1)d$, $402 = 2 + 25d$; hence, $d = 16$.

In a problem dealing with an A.P., write down the first few terms and describe them in the language of the problem. Then, decide which elements are known and which you wish to find.

Example 3. A man invests \$1000 at the end of each year for 30 years at 6% simple interest. Find the accumulated value of his investments at the end of 30 years, if no interest is withdrawn.

Solution. 1. The 1st \$1000 invested will draw interest at 6% for 29 years, or \$1740 interest; the resulting amount at the end of 30 years is \$2740. The 2d \$1000 will draw interest for 28 years, etc.

2. The total amount at the end of 30 years is

$$2740 + 2680 + 2620 + \cdots + 1060 + 1000.$$

We desire S for an A.P. where $a = 2740$, $n = 30$, $l = 1000$, and $d = -60$.

3. From $S = \frac{n}{2}(a + l)$, $\qquad S = \frac{30}{2}(2740 + 1000) = \$56{,}100.$

108. Arithmetic means. The 1st term, a, and the last term, l, in a progression are called the *extremes* of the progression. The other terms are called *means* between a and l. *To insert k arithmetic means* between two numbers, a and l, means to find a sequence of k numbers which, when placed between a and l, give rise to an A.P. with a and l as its extremes.

Example 1. Insert five arithmetic means between 13 and -11.

Solution. 1. After the means are inserted, they will complete an A.P. of seven terms, with $a = 13$ and $l = -11$. We shall find d for the progression and then form the terms.

2. From $l = a + (n - 1)d$,

$$-11 = 13 + 6d; \qquad d = -4.$$

3. Hence, the missing terms are $(13 - 4)$, or 9; $(9 - 4)$, or 5; etc. The A.P. is $(13, 9, 5, 1, -3, -7, -11)$. Therefore the arithmetic means are $(9, 5, 1, -3, -7)$.

When a *single* arithmetic mean is inserted between two numbers, it is called **the** *arithmetic mean* of the numbers. Thus, if (b, A, c) form an A.P., then A is called the arithmetic mean of b and c. Then,

$$A - b = c - A \quad \text{or} \quad 2A = c + b.$$

Hence,

$$A = \frac{b + c}{2}. \tag{1}$$

Illustration 1. The arithmetic mean of 7 and 15 is $\frac{1}{2}(7 + 15) = 11$.

EXERCISE 42

Write the first four terms of an A.P. *from the given data.*

1. $a = 15;\ d = 3.$ **2.** $a = -18;\ d = 2.$ **3.** $a = 17;\ d = -3.$

4. What do you observe if you reverse the order of the terms of the A.P. (*i*) in Problem 1; (*ii*) in the general A.P. with common difference d?

Find the last term and the sum of the A.P. *by use of formulas.*

5. 8, 13, 18, $\cdots$ to 15 terms.

6. 3, 5, 7, $\cdots$ to 41 terms.

7. 9, 6, 3, $\cdots$ to 28 terms.

8. 13, 8, 3, $\cdots$ to 17 terms.

9. 2.06, 2.02, 1.98, $\cdots$ to 33 terms.

10. 5, $4\frac{1}{2}$, 4, $\cdots$ to 81 terms.

Certain of the elements a, d, l, n, and S are given. Find the others.

11. $a = 10, l = 410, n = 26.$

12. $a = 4, l = 72, n = 18.$

13. $l = 53, d = 4, n = 19.$

14. $a = 27, l = 11, d = -\frac{1}{4}.$

15. $S = -2496, n = 52, a = 3.$

16. $S = 2337, n = 38, d = \frac{7}{2}.$

Find the value of k for which the sequence of three terms forms an A.P.

17. $(3 - 2k)$; $(2 - k)$; $(4 + 3k)$. **18.** $(2 + k)$; $(2 + 4k)$; $(5k - 1)$.

19. Insert four arithmetic means between 2 and 17.

20. Insert five arithmetic means between 7 and 19.

21. Insert six arithmetic means between 15 and -16.5.

22. Insert five arithmetic means between $-\frac{3}{2}$ and 6.

Find the arithmetic mean of the numbers.

23. 6; 38. **24.** 15; 37. **25.** -13; 27. **26.** x; y.

27. Find the 45th term in an A.P. where the 3d term is 7 and the common difference is $\frac{1}{3}$.

28. Find the 59th term in an A.P. where the 4th term is 9 and the common difference is $-.4$.

29. In the A.P. .97, 1.00, 1.03, $\cdots$, which term is 5.02?

30. In the A.P. 16, 13.5, 11, $\cdots$, which term is -129?

31. Find the common difference of an A.P. whose 6th term is 9 and 37th term is 54.

32. Find the sum of all odd integers from 3 to 209, inclusive.

33. Find the sum of all even integers from 10 to 380 inclusive.

34. Find the sum of the first 38 positive integral multiples of 3.

35. Find the sum of all positive integral multiples of 5 which are less than 503.

36. Find the sum of all positive and negative integral multiples of 6 between -55 and 363.

37. There are 17 rows of billiard balls in a symmetrical triangular arrangement on a table, with 49 balls in the first row and 3 less balls in each other row than in the one preceding it. How many balls are on the table?

38. If the 5th term of an A.P. is 430 and the 45th term is 110, find the 2d term.

39. A man invests $1000 at the end of each year for 16 years at 6% simple interest. What is the accumulated value of his investments at the end of 16 years?

40. A man invests $1000 at the beginning of each year for 25 years at 5% simple interest. Find the accumulated value of his investments at the end of 25 years.

41. The 4th term of an A.P. is 215 and the 44th term is 55. Find the sum of the first 20 terms.

Find the total sum of money paid by the debtor in discharging his debt.

42. Debtor borrows $15,000. *Agrees to pay:* at the end of each year for 10 years, $1500 principal and simple interest at 3% on all principal outstanding during the year.

43. Debtor borrows $30,000. *Agrees to pay:* at the end of each year for 30 years, $1000 principal and simple interest at 5% on all principal outstanding during the year.

109. A geometric progression (abbreviated G.P.) is a sequence of numbers called *terms*, each of which, after the first, is obtained by *multiplying* the preceding term by a fixed number called the **common ratio.** The common ratio equals the *ratio of any term, after the first, to the one preceding it.* To determine whether or not a sequence of numbers forms a G.P., we *divide* each number by the one which precedes it. All of these ratios are equal if the terms form a G.P.

ILLUSTRATION 1. In the G.P. $16, -8, +4, -2, \cdots$, the common ratio is $-\frac{1}{2}$; the 5th term would be $(-\frac{1}{2})(-2) = +1$.

110. The nth term and the sum of a G.P. Let a be the 1st term and r be the common ratio. Then, the 2d term is ar; the 3d term is ar^2. In each of these terms the exponent of r is 1 *less than the number of the term.* Similarly, the 8th term is ar^7. The nth term is the $(n-1)$th after the 1st and hence is found by multiplying a by $(n-1)$ factors r, or by r^{n-1}. Hence, if l represents the nth term,

$$l = ar^{n-1}. \tag{1}$$

ILLUSTRATION 1. If $a = 3$ and $r = 2$, the 7th term is $3(2^6) = 192$.

Let S be the sum of the first n terms of the G.P. The nth term is ar^{n-1}, the $(n-1)$th term is ar^{n-2}, etc. Hence,

$$S = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}, \tag{2}$$

and

$$Sr = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n; \tag{3}$$

in (3) we multiplied both sides of (2) by r. On subtracting each side of (3) from the corresponding side of (2), we obtain

$$S - Sr = a - ar^n, \tag{4}$$

because each term, except ar^n, on the right in (3) cancels a corresponding term in (1). From (4), $S(1-r) = a - ar^n$, or

$$S = \frac{a - ar^n}{1 - r}. \tag{5}$$

Since $l = ar^{n-1}$, then $rl = ar^n$. Hence, from (5),

$$S = \frac{a - rl}{1 - r}. \tag{6}$$

EXAMPLE 1. Find the sum of the G.P. $2, 6, 18, \cdots$ to six terms.

SOLUTION. $n = 6$; $a = 2$; $r = 3$. From (5),

$$S = \frac{2 - 2\cdot 3^6}{1 - 3} = \frac{2 - 1458}{-2} = 728.$$

EXAMPLE 2. Find the sum of the geometric progression

$$(1.05)^2 + (1.05)^5 + (1.05)^8 + \cdots + (1.05)^{35}.$$

SOLUTION. $a = (1.05)^2$; $r = (1.05)^3$; $l = (1.05)^{35}$. From formula 6, which is convenient when l is explicitly given,

$$S = \frac{(1.05)^2 - (1.05)^3(1.05)^{35}}{1 - (1.05)^3} = \frac{(1.05)^2 - (1.05)^{38}}{1 - (1.05)^3}.$$

Note 1. When a sufficient number of the *elements* (a, r, n, l, S) are given, we find the others by use of the fundamental formulas 1, 5, and 6.

EXAMPLE 3. If $S = 750$, $r = 2$, and $l = 400$, find n and a.

SOLUTION. 1. From $S = \dfrac{a - rl}{1 - r}$, $750 = \dfrac{a - 800}{1 - 2}$; hence, $a = 50$.

2. From $l = ar^{n-1}$, $\qquad 400 = 50(2^{n-1})$; $2^{n-1} = \dfrac{400}{50} = 8$;

$$2^{n-1} = 2^3; \quad \text{hence,} \quad n - 1 = 3, \quad \text{or} \quad n = 4.$$

EXAMPLE 4. Insert two geometric means between 6 and $\frac{16}{9}$.

SOLUTION. After the means are inserted, they will complete a G.P. of four terms with $a = 6$ and $l = \frac{16}{9}$. We shall find the common ratio of the progression, and then its two middle terms. From $l = ar^{n-1}$, with $n = 4$,

$$\tfrac{16}{9} = 6r^3; \quad r^3 = \tfrac{8}{27}; \quad r = \sqrt[3]{\tfrac{8}{27}} = \tfrac{2}{3}.$$

The G.P. is $(6, 4, \frac{8}{3}, \frac{16}{9})$. The geometric means are 4 and $\frac{8}{3}$.

Comment. We shall always desire only *real-valued* means.

EXERCISE 43

Write the first four terms of a G.P. *for the given data.*

1. $a = 5$; $r = 3$. **2.** $a = 4$; $r = -2$. **3.** $a = 16$; $r = \frac{1}{2}$.

In case the numbers form a G.P., *state its common ratio and write two more terms of the* G.P.

4. $4, -12, 36, 108$. **5.** $8, \frac{8}{3}, \frac{8}{9}, \frac{8}{27}$. **6.** $(1.01)^{-5}, (1.01)^{-3}, (1.01)^{-1}$.

7. What fact do you observe if we reverse (*i*) the G.P. (81, 27, 9, 3); (*ii*) the G.P. with 1st term a and common ratio r?

By use of $l = ar^{n-1}$, *find the specified term of the given* G.P.

8. 3, 9, 27; find the 6th term. **9.** $4, -12, 36$; find the 9th term.

Find the last term and the sum of the G.P.

10. $3, 6, 12, \cdots$ *to* 7 *terms.*

11. $10, 5, \frac{5}{2}, \cdots$ *to* 6 *terms.*

12. $50, 5, .5, \cdots$ *to* 7 *terms.*

13. $-8, 4, -2, \cdots$ *to* 5 *terms.*

14. $\frac{1}{25}, -\frac{1}{5}, 1, \cdots$ *to* 6 *terms.*

15. $a, ax^3, ax^6, \cdots$ *to* 15 *terms.*

Find an expression for the sum of each G.P. *by use of* (6), *page* 107. *Simplify by use of the laws of exponents when possible.*

16. $2 + 1 + \cdots + \frac{1}{256}$.

17. $6 + 18 + \cdots + 4374$.

18. $1 + (1.03) + (1.03)^2 + \cdots + (1.03)^{40}$.

19. $(1.06)^4 + (1.06)^5 + (1.06)^6 + \cdots + (1.06)^{29}$.

20. $1 + (1.02)^3 + (1.02)^6 + \cdots$ to 21 terms.

21. $(1.02)^{-15} + (1.02)^{-14} + (1.02)^{-13} + \cdots + (1.02)^{-1}$.

22. $(1.03)^{-16} + (1.03)^{-14} + (1.03)^{-12} + \cdots + (1.03)^{-4}$.

23. $1 + (1.02)^{\frac{1}{2}} + (1.02) + (1.02)^{\frac{3}{2}} + \cdots + (1.02)^{\frac{19}{2}}$.

Find the missing elements of the G.P.

24. $a = 5;\ r = 2;\ l = 320$.

25. $a = 2;\ r = 3;\ l = 162$.

26. $r = 10;\ a = .001;\ l = 1000$.

27. $S = 275;\ r = -2;\ l = 400$.

28. $S = \frac{910}{9};\ a = -\frac{5}{9};\ l = 135$.

29. $a = 256;\ r = \frac{1}{2};\ l = \frac{1}{4}$.

Find the specified term without finding the first term of the G.P.

30. The 10th term, if the 6th term is 5 and common ratio is 2.

31. The 5th term, if the 9th term is 80 and common ratio is $\frac{1}{2}$.

Insert the specified number of geometric means.

32. Five, between 2 and 128.

33. Three, between 4 and 324.

34. Four, between $\frac{1}{3}$ and 81.

35. Six, between .1 and 1,000,000.

36. If (x, G, y) form a G.P., prove that $G^2 = xy$.

If x and y are of the same sign, and if a single geometric mean G of the same sign is inserted between x and y, then G is called **the geometric mean** *of x and y; (x, G, y) form a* G.P. *Find the geometric mean of the numbers.*

37. $x;\ y$.

38. $\frac{1}{9};\ 36$.

39. $4;\ 25$.

40. $-9;\ -25$.

41. If the 6th term of a G.P. is 5 and the 10th term is $\frac{5}{16}$, find the intermediate terms.

42. For what values of k do the three quantities $(k + 3)$, $(6k + 3)$, and $(20k + 5)$ form a G.P.?

43. Find the sum of a G.P. of 7 terms whose 3d term is $\frac{5}{4}$ and 6th is $\frac{5}{32}$.

44. How many ancestors have you had in the 10 preceding generations if no ancestor appears in more than one line of descent?

45. An investment paid a man, in each year after the first, twice as much as in the preceding year. If his investment paid him \$6750 in the first four years, how much did it pay in the first and the fourth years?

46. Find the sum of the first 19 positive integral powers of 1.07, given that $(1.07)^{10} = 1.967$.

MISCELLANEOUS EXERCISE 44

1. At a bazaar, tickets are marked with the consecutive odd integers 1, 3, 5, $\cdots$ and are drawn at random by those entering. If each person pays as many cents as the number on his ticket, how much money is received if 1000 tickets are sold?

2. A rubber ball is dropped from a height of 100 feet. On each rebound, the ball rises $\frac{1}{2}$ of the height from which it last fell. What distance has the ball traveled up to the instant the ball hits the ground for the 8th time?

3. In a potato race, 20 potatoes are placed at intervals of 5 feet in a line from the starting point, with the nearest potato 25 feet away. A runner is required to bring the potatoes back to the starting place one at a time. How far would he run in bringing in all the potatoes?

4. If \$605 is to be divided among 11 men so that the first one receives \$5 and each succeeding man obtains a fixed amount more than the preceding man, how much will the 10th man receive?

5. In a professional golf tournament, the total prize money of \$5155 is divided among the 5 best men so that each, except the fifth best, receives $2\frac{1}{2}$ times as much as the man who ranks just below him. How much does each of the five receive?

6. Find an expression for the sum of the first n positive odd integers.

7. Find the sum of the first 40 positive integral powers of x.

8. In creating a vacuum in a container, a pump draws out $\frac{1}{4}$ of the remaining air at each stroke. What part of the original air has been removed by the end of the 7th stroke?

9. At the beginning of each year, a man invests \$400 at simple interest at the rate 4%. At the end of 20 years, what is the accumulated value of his investments, if none of them have been disturbed since they were made?

10. A mountain climber ascends 1000 feet in the first hour, and 100 feet less in each succeeding hour than in the preceding hour. When will he be 5400 feet above his starting point?

11. A pendulum bob moves over a path 15 inches long on the first swing from its extreme position on one side to the extreme on the other side. In each succeeding swing the bob travels $\frac{4}{5}$ of the distance of the preceding swing. How far does the bob travel during the first 6 swings?

12. A speculator will make \$1200 during the first month and, thereafter, in each month, \$100 less than in the preceding month. If his original capital is \$4200, when will he become bankrupt?

13. Prove that the squares of the terms of a G.P. also form a G.P. Then state a more general theorem of this nature.

14. Find an A.P. of four numbers whose sum is 24 if the sum of their squares is 164.

15. Prove that the reciprocals of the terms of a G.P. also form a G.P.

16. The radiator of a motor truck contains 10 gallons of water. We draw off 1 gallon and replace it with alcohol; then, we draw off 1 gallon of the mixture and replace it by alcohol; etc., until 7 drawings and replacements have been made. How much alcohol is in the final mixture?

17. In a certain positive integral number of three digits, the digits form an A.P. and their sum is 15. If the digits are reversed, the new number is 594 less than the original number. Find the original number.

18. If $y = 5x + 8$, find the sum of the values of y corresponding to the successive integral values $x = 1, 2, 3, \cdots, 30$.

★19. Prove that if $y = mx + b$, where m and b are constants, the values of y corresponding to $x = h, 2h, 3h, \cdots$ form an A.P., for any value of h.

★20. Prove that the average value of the terms of an A.P. of n terms is $\frac{1}{2}(a + l)$, where the *average value* means *the sum divided by n*.

Note 1. If P is the value of a certain quantity *now*, and if its value increases at the rate i (expressed as a decimal) per year, then the new value at the end of one year is $P + Pi$, or $P(1 + i)$. That is, *the value at the end of any year is* $(1 + i)$ *times the value at the end of the last year.* The values at the ends of the years form a G.P. whose common ratio is $(1 + i)$. If F represents the value at the end of n years, then

$$F = P(1 + i)^n.$$

This formula is referred to as the **compound interest law** because, if a principal P is invested now at the rate i, compounded annually, the amount F at the end of n years will be $P(1 + i)^n$. In all of the following problems, it will be assumed that any *rate* is *constant*.

★21. If 5000 units of a commodity are consumed in the 1st year, and if the annual rate of increase in consumption is 8%, find the total consumption during the first 15 years, given that $(1.08)^{15} = 3.17216911$.

★22. The number of inhabitants in a city increased from 128,000 to 432,000 in six years. Find the annual rate of increase.

★23. A piece of ground was purchased 8 years ago for \$16,200; its value now is \$51,200. Find the rate per year at which the value of the ground has increased.

★24. The value of a certain quantity *decreases* at the rate w (expressed as a decimal) per year. If H is the value now, and K is the value at the end of n years, prove that $K = H(1 - w)^n$.

★25. A motor truck was purchased for \$2500, and its value 4 years later is \$1024. Find the rate per year at which the value has decreased.

★26. A hotel, purchased 3 years ago for \$512,000, is sold for \$343,000. Find the rate per year at which its value has decreased.

★111. Harmonic progressions. A sequence of numbers is said to form a *harmonic* progression* if their reciprocals form an *arithmetic progression.*

ILLUSTRATION 1. The sequence $(1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9})$ is a harmonic progression because the reciprocals (1, 3, 5, 7, 9) form an A.P.

To insert k harmonic means between two numbers, we first insert k *arithmetic means* between the *reciprocals of the numbers.* The reciprocals of the arithmetic means are the harmonic means.

EXAMPLE 1. Insert five harmonic means between 4 and 16.

SOLUTION. 1. First, we insert 5 arithmetic means between $\frac{1}{4}$ and $\frac{1}{16}$.

2. From $l = a + (n - 1)d$, with $a = \frac{1}{4}$, $l = \frac{1}{16}$, and $n = 7$, we find

$$\tfrac{1}{16} = \tfrac{1}{4} + 6d; \quad d = -\tfrac{1}{32}.$$

3. Hence, the A.P. is $(\frac{1}{4}, \frac{7}{32}, \frac{6}{32}, \frac{5}{32}, \frac{4}{32}, \frac{3}{32}, \frac{1}{16})$.

4. The corresponding harmonic progression is $(4, \frac{32}{7}, \frac{16}{3}, \frac{32}{5}, 8, \frac{32}{3}, 16)$. Hence, the harmonic means are $(\frac{32}{7}, \frac{16}{3}, \frac{32}{5}, 8, \frac{32}{3})$.

If (c, H, d) form a harmonic progression, then H is called **the harmonic mean** of c and d.

★EXERCISE 45

Insert the specified number of harmonic means.

1. Four, between $\frac{1}{2}$ and $\frac{1}{12}$.
2. Five, between $\frac{1}{4}$ and $\frac{1}{28}$.
3. Four, between $\frac{5}{14}$ and $\frac{5}{4}$.
4. Four, between 4 and 24.
5. Five, between $\frac{5}{3}$ and $\frac{1}{3}$.
6. Four, between $-\frac{1}{3}$ and 3.

Find the harmonic mean of the numbers.

7. 4; $\frac{4}{3}$. **8.** 9; 6. **9.** 4; -8. **10.** 12; 36. **11.** 6; -18.

12. Derive a formula for the harmonic mean of x and y.

13. If A, G, and H represent, respectively, the arithmetic, geometric, and harmonic means of x and y, prove that G is the geometric mean of A and H; that is, show that $G^2 = AH$.

★112. Geometric progressions with infinitely many terms. Let S_n represent the sum of the progression $a, ar, ar^2, \cdots, ar^{n-1}$. Then, by (5), page 107,

* Suppose that a set of strings of the same diameter and substance are stretched to uniform tension. If the lengths of the strings form a harmonic progression, a harmonious sound results if two or more strings are caused to vibrate at one time. This fact accounts for the name *harmonic progression.*

$$a + ar + ar^2 + \cdots + ar^{n-1} = S_n = \frac{a}{1-r} - \frac{ar^n}{1-r}. \tag{1}$$

ILLUSTRATION 1. Consider the endless geometric progression

$$1,\ \frac{1}{2},\ \frac{1}{4},\ \cdots,\ \frac{1}{2^{n-1}},\ \cdots\ \textit{to infinitely many terms.} \tag{2}$$

In (2), $\quad r = \frac{1}{2}; \quad$ the nth term is $\frac{1}{2^{n-1}}; \quad 1 - r = \frac{1}{2}; \quad ar^n = \frac{1}{2^n}.$

By (1), $$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}} = S_n = 2 - \frac{1}{2^{n-1}}. \tag{3}$$

If n grows larger, without limit, the term $\frac{1}{2^{n-1}}$ grows smaller, and is as near to zero as we please, if n is sufficiently large. Thus, if $n = 65$,

$$\frac{1}{2^{n-1}} = \frac{1}{2^{64}} = \frac{1}{18{,}446{,}744{,}073{,}709{,}551{,}616},$$

which is practically zero. Hence, in (3), S_n will be as near to $(2 - 0)$ as we please for all values of n which are sufficiently large. To summarize this statement we say that *as n becomes infinite S_n approaches the* **limit** 2, and we call 2 the *sum* of the progression $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$ *to infinitely many terms.* We sometimes use "$n \to \infty$" to abbreviate "*n becomes infinite.*" Then, our conclusion can be briefly written

$$\mathop{limit}_{n\to\infty} S_n = 2.$$

Now, consider ($a, ar, ar^2, \cdots$ *to infinitely many terms*), under the condition that r is a number between -1 and $+1$. Then, as $n \to \infty$ the absolute value of the numerator ar^n in (1) grows smaller, and is as near to zero as we please for all values of n sufficiently large. Hence, from (1) we see that, as n grows large without limit, the value of S_n approaches

$$\left(\frac{a}{1-r} - \frac{0}{1-r}\right), \qquad \text{or} \qquad \frac{a}{1-r}.$$

In other words,

$$\mathop{limit}_{n\to\infty} S_n = \frac{a}{1-r}. \tag{4}$$

This limit of the sum of n terms, as n becomes infinite, is called the **sum** of the geometric progression with infinitely many terms. If S represents this sum, then

$$S = \frac{a}{1-r}. \tag{5}$$

Note 1. Recognize that S in (5) is *not* a sum in the ordinary sense of the word, but is *the limit of the sum of n terms as n grows large without bound.* We may summarize our result compactly as follows, remembering that the left member below is *not* a sum in the ordinary sense: **if $|r| < 1$,**

$$(a + ar + ar^2 + \cdots \textit{ to infinitely many terms}) = \frac{a}{1 - r}. \tag{6}$$

ILLUSTRATION 2. By use of (6), with $a = 5$ and $r = \frac{1}{2}$,

$$\left(5 + \frac{5}{2} + \frac{5}{4} + \cdots \textit{ to infinitely many terms}\right) = \frac{5}{1 - \frac{1}{2}} = 10.$$

Practically, this means that, if we should add a relatively large number of terms of the progression, we would obtain approximately 10, and that by adding enough terms we can obtain a result as close to 10 as we may desire. Thus, $S_{11} = 9\frac{1019}{1024}$.

The indicated sum of a sequence of numbers is frequently called a **series.** An expression of the form

$$u_1 + u_2 + u_3 + \cdots \textit{ to infinitely many terms} \tag{7}$$

is called an **infinite series.** Accordingly, the expression on the left in (6) is referred to as the *infinite geometric series.*

EXAMPLE 1. Find a rational fraction equal to the endless repeating decimal 5818181 $\cdots$.

SOLUTION. Because of the meaning of the decimal notation,

$$.5818181 \cdots = .5 + .081 + .00081 + \cdots \textit{ to inf. many terms},$$

where we notice that $(.081 + .00081 + \cdots)$ is an infinite geometric series with $a = .081$, and $r = .01$. By (6), the sum of this series is

$$\frac{.081}{1 - .01} = \frac{.081}{.99} = \frac{9}{110}.$$

Hence, $$.5818181 \cdots = .5 + \tfrac{9}{110} = \tfrac{5}{10} + \tfrac{9}{110} = \tfrac{32}{55}.$$

Comment. By use of the method of Example 1, we can show that *any endless repeating decimal is a rational number.*

In any infinite series such as (7), let S_n represent the sum of the first n terms. Then we say that the series *has a sum S,* and call the series a **convergent infinite series** which *converges* to S in case *the limit of S_n is S as n becomes infinite.* If S_n has no limit as n becomes infinite, we say that the infinite series is **divergent,** or *diverges.**

* A detailed consideration of convergent and divergent series is given at a more advanced stage.

Note 2. In this section we have proved that the infinite geometric series in parentheses in (6) has a sum, or *converges*, when $|r| < 1$. When $|r| \geqq 1$, the series is divergent, or *does not have a sum*, because in this case S_n in (1) does not approach a limit as $n \to \infty$. Thus, for the G.P. $(1, 2, 4, \cdots)$ where $r = 2$, we find that S_n increases beyond all bounds as $n \to \infty$.

★EXERCISE 46

Find the sum of each of the following infinite geometric series.

1. $6 + 2 + \frac{2}{3} + \cdots$.
2. $16 + 4 + 1 + \cdots$.
3. $5 + .5 + .05 + \cdots$.
4. $1 - \frac{1}{4} + \frac{1}{16} - \cdots$.
5. $1 - .1 + .01 - .001 + \cdots$.
6. $.16 - .016 + .0016 - \cdots$.

Find a rational number which equals the given endless repeating decimal, where repeating parts are written three times.

7. $.666 \cdots$.
8. $.555 \cdots$.
9. $.131313 \cdots$.
10. $.252525 \cdots$.
11. $.010101 \cdots$.
12. $.1666 \cdots$.
13. $5.222 \cdots$.
14. $4.090909 \cdots$.
15. $3.454545 \cdots$.
16. $14.212121 \cdots$.
17. $.589111 \cdots$.
18. $8.48484 \cdots$.
19. $363.636 \cdots$.
20. $14.3143143 \cdots$.
21. $73.03030 \cdots$.
22. $.142857142857142857 \cdots$.
23. $.076923076923076923 \cdots$.

24. A pendulum is being brought to rest by air resistance; the path of each swing (after the first) of the pendulum bob is .95 as long as that of the previous swing. If the path of the first swing is 30 inches long, find the total distance traveled by the bob in coming to rest.

25. A rubber ball is dropped from a height of 40 feet. On each rebound the ball rises to $\frac{3}{4}$ of the height from which it last fell. Find the distance traversed by the ball in coming to rest.

26. The side of a certain square is 10 inches long. A second square is drawn by connecting the mid-points of the sides of the 1st square; a 3d square is drawn by connecting the mid-points of the sides of the 2d square; etc., without end. Find the sum of the areas of all the squares of which this process conceives.

Note 1. If $|r| < 1$, we know that S, of (5) on page 113, is approximately equal to S_n if n is large, and our confidence in this approximation increases as n increases. When n is large, it is decidedly easier to compute S than S_n, and hence it is convenient at times to use S in place of S_n.

27. Find S_{10} for the G.P. $(3, \frac{3}{2}, \frac{3}{4}, \cdots)$. Also, find the sum of this progression extended to infinitely many terms.

28. Find S_{139}, approximately, for the G.P. $(3, \frac{3}{4}, \frac{3}{16}, \cdots)$.

CHAPTER TEN

Inequalities

113. Inequalities. If a and b are *real** numbers, we say that a is *greater than* b, or that b is *less than* a, in case $a - b$ is *positive,* and we write $a > b$, or $b < a$. If $b < a$ and if a and b are plotted in Figure 12, then b is to the *left* of a. Any relation expressed by use of $<$ or $>$ is called an *inequality.*

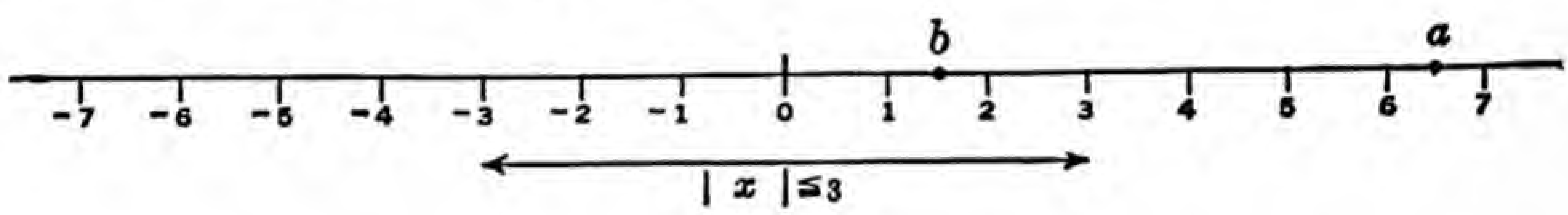


FIG. 12

ILLUSTRATION 1. The expression $1 < x < 4$ is read "1 is less than x and x is less than 4." These simultaneous conditions on x state that, in Figure 12, the value of x would lie on the interval *between* 1 *and* 4.

ILLUSTRATION 2. The expression $|x| \leqq 3$ is read "the absolute value of x is less than or at most equal to 3." Then, x lies *between* -3 *and* 3 *inclusive.* Hence,

$$\textit{if} \quad |x| \leqq 3 \quad \textit{then} \quad -3 \leqq x \leqq 3.$$

If an inequality does not involve literal numbers or if it is true for all values of the letters involved, we call it an **absolute inequality.** A **conditional inequality** is one which is true only for certain values of the letters involved. **To solve an inequality** means to obtain a simple description of the values of the literal numbers which satisfy the inequality.

ILLUSTRATION 3. $8 > 3$ is an absolute inequality. $x - 3 < 0$ is a conditional inequality which is true only if $x < 3$. $d^2 \geqq 0$ is an absolute inequality because, for all values of d, d^2 is positive or zero.

114. Properties of inequalities. Two inequalities are said to have the *same sense* if their inequality signs point in the same direction. Thus, $A < B$ and $C < D$ have the same sense.

* The relation $a > b$ is defined for *real* numbers. All numbers in this chapter will be *real.*

Property I. *The sense of an inequality $B < A$ is not changed if the same number is added to both sides.*

Proof. 1. If C is any number, we must prove that

$$B + C < A + C.$$

2. Let $A - B = p$, which is positive because $B < A$. Hence,

$$(A + C) - (B + C) = A - B = p,$$

or $[(A + C) - (B + C)]$ is positive; therefore, $B + C < A + C$.

To subtract a number N, we add $-N$. Hence, from Property I, it follows that the sense of an inequality is not changed by subtracting the same number from both sides.

Note 1. Property I justifies the statement that the following mechanical operations on an inequality yield equivalent inequalities.

A term appearing on both sides of an inequality may be **canceled** *by subtracting the term from both sides.*

A term may be **transposed** *from one side of an inequality to the other with the sign of the term changed, by subtracting the term from both sides.*

Property II. *The sense of an inequality is not changed by multiplying both sides by the same positive number.*

Proof. 1. If $A > B$ and if k is any *positive* number, we wish to prove that $kA > kB$.

2. Let $A - B = p$, which is positive because $A > B$. Hence, $kA - kB = k(A - B) = kp$, which is positive. Therefore, $kA > kB$.

Property III. *The sense of an inequality is reversed if both sides are multiplied by the same negative number. In particular, if both sides are multiplied by -1, the inequality sign must be reversed.*

Proof. 1. If k is any *positive* number, and if $A > B$, we wish to prove that $-kA < -kB$.

2. Let $A - B = p$; then p is positive and

$$(-kB) - (-kA) = k(A - B) = kp,$$

which is positive. Hence, $-kA < -kB$.

Illustration 1. We know that $-2 < 5$. On multiplying both sides by -1 we obtain $2 > -5$, which is true.

Similarly, as a special case of Property II, the sense of an inequality is not changed by dividing both sides by the same *positive* number.

By Property III, the sense is reversed if both sides are divided by the same *negative* number.

115. Linear inequalities. An inequality is said to be *linear* in x if each side is linear in x. By use of Properties I, II, and III, we can solve any linear inequality. The procedure is like that used in solving a linear equation except that the inequality sign must be reversed if both sides are multiplied or divided by a negative number.

EXAMPLE 1. Solve: $\frac{7}{3}x - 1 < 17 - \frac{2}{3}x.$

SOLUTION. 1. Multiply both sides by 3: $7x - 3 < 51 - 2x.$

2. Add $(3 + 2x)$ to both sides: $9x < 54.$

3. Divide by 9: $x < 6.$

116. Graphical solution of an inequality. Any inequality in one variable, x, can be placed in the form $f(x) > 0$, or $f(x) < 0$, by transposing terms.

ILLUSTRATION 1. $-3x - 5x^2 < 3 + 5x$ becomes $0 < 5x^2 + 8x + 3.$

To solve an inequality $f(x) > 0$ graphically, draw a graph of $f(x)$. Then, the values of x for which the graph is above the x-axis are the values of x for which $f(x) > 0$. The values of x for which the graph is below the x-axis satisfy $f(x) < 0$.

EXAMPLE 1. Solve graphically: $0 > 7x + 4 - 2x^2.$

SOLUTION. 1. Transpose terms to give x^2 a positive coefficient:

$$2x^2 - 7x - 4 > 0.$$

2. Let $f(x) = 2x^2 - 7x - 4$. A graph of $f(x)$ is obtained in Figure 13 by use of the following values:

If $x =$	-2	$-\frac{1}{2}$	2	4	5
$f(x) =$	18	0	-10	0	11

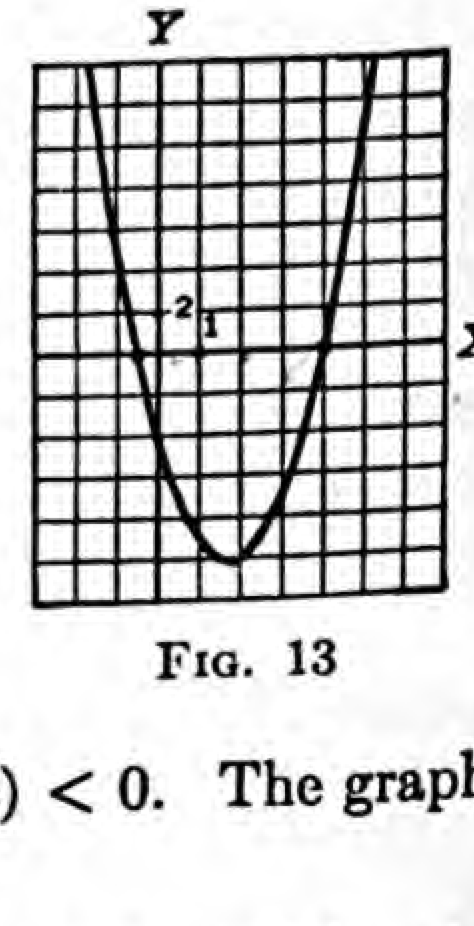


FIG. 13

3. From the graph we see that

$$f(x) > 0 \quad \textit{if} \quad x > 4 \quad \textit{and if} \quad x < -\tfrac{1}{2}.$$

Comment 1. By use of Figure 13, we can also solve $f(x) < 0$. The graph is *below* the x-axis if $-\frac{1}{2} < x < 4.$

Comment 2. In Figure 13, we needed *accurately* the points where $f(x) = 0$. Hence, before graphing we solved $2x^2 - 7x - 4 = 0$, whose roots are $x = -\frac{1}{2}$ and $x = 4$.

EXERCISE 47

Find the values of x for which the inequality is true.

1. $4x - 24 > 0.$
2. $15 - 6x < 0.$
3. $3x - 12 < 24.$
4. $4 + 3x < 10.$
5. $\frac{2}{3}x + 4 > 2.$
6. $\frac{3}{4}x + 5 < 7.$
7. $5x - 5 > -9 + 3x.$
8. $2x + 18 < 3 - 3x.$
9. $5 - 7x > 10 - 10x.$
10. $2x - \frac{4}{3} < \frac{1}{5}x - 2.$
11. $5x - \frac{3}{5} < 2 - 4x.$
12. $3x - \frac{1}{5} < \frac{5}{3}x + 3.$

Think of x as plotted on the scale in Figure 12 and express the given fact by use of inequalities without an absolute value symbol.

13. x lies to the right of 5.
14. x lies to the left of -4.
15. x lies to the left of 3.
16. x lies to the right of -2.
17. x lies between 2 and 9.
18. x lies between -4 and 4.
19. $|x| < 5.$
20. $|x| < 2.$
21. $|x| > 3.$
22. $|x| < a.$

State by use of an absolute value symbol and describe where x would lie on the scale in Figure 12.

23. $-6 < x < 6.$
24. $-8 < x < 8.$
25. $x < -7$ or $x > 7.$

Solve graphically and state results by use of inequalities.

26. $7x - 5 < 0.$
27. $2 - 5x > 0.$
28. $x^2 - 9 < 0.$
29. $25 < x^2.$
30. $x^2 - 6x + 8 > 0.$
31. $x^2 - 2x - 8 < 0.$
32. $2x^2 + 5x < 3.$
33. $4 + 3x < x^2.$
34. $5 - 3x < 2x^2.$
35. $4x^2 + x > 0.$
36. $2x - 5x^2 > 0.$
37. $9x^2 + 16 < 0.$
38. $10{,}000x^2 > 49.$
39. If $x < y$ and $y < z$, prove that $x < z$.

Solve by inspection, without graphing.

40. $x^2 < 144.$
41. $x^2 > 100.$
42. $x^2 < a^2$, where $a > 0.$

For what values of x is the radical real?

43. $\sqrt{x^2 - 16}.$
44. $\sqrt{4 - x^2}.$
45. $\sqrt{x^2 - 5x + 6}.$

★*Find the values of k for which the roots of the equation in x are real, and those values for which the roots are imaginary.*

46. $2kx^2 - 4x + 3 = 0.$
47. $x^2 = 2kx - 4.$
48. $kx^2 + 4k = 3x.$

HINT. Compute the discriminant.

★*Find the values of k for which the graphs of the equations (a) are tangent; (b) meet in no points; (c) meet in two points.*

49. $\begin{cases} x^2 + y^2 = 16, \\ y = kx + 5. \end{cases}$
50. $\begin{cases} 2x^2 + y^2 = 6, \\ y = kx + 3. \end{cases}$
51. $\begin{cases} 4x^2 + 2y^2 = 3, \\ y = kx + 3. \end{cases}$

★**117. Analytical proofs of inequalities.** To establish a specified inequality under given conditions about the variables, it is sometimes convenient to proceed as follows.

1. SUGGESTIVE PART. *Assume that the inequality is true and, from it, proceed to some simpler inequality which is easily verified.*

2. DEMONSTRATION. *Start with this simpler inequality and, from it, derive the given inequality by* **reversing** *the acts of Step 1, where the possibility of each reversal must be justified.*

EXAMPLE 1. Prove that, if $x \neq 1$ and $x > 0$, $\qquad \dfrac{1}{x} + x > 2.$

SOLUTION. 1. *Suggestive part.* IF the inequality is true, then

(multiply by x) $\qquad 1 + x^2 > 2x;$

(subtract $2x$) $\qquad 1 - 2x + x^2 > 0, \quad$ or $\quad (1 - x)^2 > 0.$

2. *Proof.* Since x is real and $\neq 1$, hence $1 - x \neq 0$ and

$$(1 - x)^2 > 0, \quad \text{or} \quad 1 - 2x + x^2 > 0;$$

(add $2x$) $\qquad 1 + x^2 > 2x.$

Since $x > 0$, by Property II we may divide both sides above by x without altering the inequality and thus obtain $\dfrac{1}{x} + x > 2.$ Q.E.D.

★EXERCISE 48

1. If $x > 0$ and $y > 0$, prove that $\dfrac{x}{y} + \dfrac{y}{x} > -2.$

2. If $x + y > 0$ and $x \neq y$, prove that $\dfrac{x + y}{2} > \dfrac{2xy}{x + y}.$

If $x > 0$, $y > 0$, and $x \neq y$, prove the inequality.

3. $\dfrac{x + y}{2} > \sqrt{xy}.$ $\qquad$ 4. $\dfrac{x}{y} + \dfrac{y}{x} > 2.$ $\qquad$ 5. $\dfrac{2xy}{x + y} < \sqrt{xy}.$

Note 1. From Problems 3 and 5, it follows that, if A, G, and H are, respectively, the arithmetic, geometric, and harmonic means of x and y, then $H < G < A$.

If $x > 0$, $y > 0$, and $x > y$, prove the inequality.

6. $x^3 - y^3 > (x - y)^3.$ $\qquad$ 7. $x^3 - y^3 > xy(y - x).$ $\qquad$ 8. $x^3 > y^3.$

9. If $x > 1$, prove that $x < x^2$, without graphing.

10. If $0 < x < 1$, prove that $x > x^3$, without graphing.

11. If $1 < x < 2$, prove that $(x - 1)(x - 2)(x - 3) > 0.$

12. Prove that $a^2 + b^2 \geqq 2ab$, for all values of a and b.

CHAPTER ELEVEN

Complex Numbers

118. Complex numbers.* If we let i represent the symbol $\sqrt{-1}$, with the property that $i^2 = -1$, and if a and b are real, then we call $a + bi$ a *complex number* whose *real* part is a and *imaginary* part is bi; we call b the *coefficient of the imaginary part*. Any real number is considered as a complex number in which the coefficient of the imaginary part is *zero*. A complex number $a + bi$ is called an *imaginary number* if $b \neq 0$. If $a = 0$ and $b \neq 0$, then $a + bi$ is called a *pure imaginary number*.

If P is positive, we let the symbol $\sqrt{-P}$ represent $i\sqrt{P}$; then, $(-P)$ has the two square roots $\pm\sqrt{-P}$, or $\pm i\sqrt{P}$.

ILLUSTRATION 1. $\sqrt{-16} = i\sqrt{16} = 4i$.

ILLUSTRATION 2. If $a > 0$ then $\sqrt{-9a^2} = 3ai$.

ILLUSTRATION 3. In the form of a complex number, $6 = 6 + 0i$.

DEFINITION I. *Two complex numbers $a + bi$ and $c + di$ are called equal in case $a = c$ and $b = d$.*

Since $0 = 0 + 0i$, it follows from Definition I that

$$\text{if} \quad a + bi = 0, \quad \text{then} \quad a = 0 \quad \text{and} \quad b = 0. \tag{1}$$

We agree that,† in any indicated sum, product, or quotient involving integral powers of complex numbers, the result shall be that which is obtained by operating as if i were a literal number subject to the rules of algebra as they apply to real numbers. At each stage of any algebraic operation involving complex numbers, each positive integral power of i should be expressed properly as 1, -1, i, or $-i$, by use of the fact that $i^2 = -1$.

Note 1. In this chapter, i always represents $\sqrt{-1}$; all other literal numbers represent real numbers and any literal number in a radicand will represent a positive number, unless otherwise specified.

* The student should review Section 71, page 64.

† For a logical foundation of the algebra of complex numbers, see DICKSON'S *Elementary Theory of Equations*, page 21.

ILLUSTRATION 4. In accordance with the preceding agreement, we have defined the left member in each of the following equations to represent the expression given in the right member.

$$(a + bi) + (c + di) = a + c + (b + d)i.$$
$$(a + bi)(c + di) = ac + (bc + ad)i + bdi^2, \text{ or}$$
$$(a + bi)(c + di) = (ac - bd) + (bc + ad)i.$$

ILLUSTRATION 5. $(3 - 5i)^2 = 9 - 30i + 25i^2 = -16 - 30i.$

$$\sqrt{-45}\sqrt{-5} = i\sqrt{45} \cdot i\sqrt{5} = 3i^2\sqrt{5}\sqrt{5} = -15.$$

119. Conjugate complex numbers. If two complex numbers differ only in the signs of their imaginary parts, the two numbers are called *conjugate complex numbers,* and either is called the *conjugate* of the other.

ILLUSTRATION 1. The conjugate of $(a + bi)$ is $(a - bi)$. Moreover,

$$(a + bi) - (a - bi) = 2bi; \tag{1}$$
$$(a + bi) + (a - bi) = 2a; \tag{2}$$
$$(a + bi)(a - bi) = a^2 - b^2i^2 = a^2 + b^2. \tag{3}$$

Notice that the *difference* of two conjugates is a *pure imaginary* number, whereas their *sum* and their *product* are *real* numbers.

A quotient of two complex numbers can be reduced to the form $a + bi$ by multiplying both numerator and denominator by the conjugate of the denominator.

ILLUSTRATION 2.
$$\frac{5 + 2i}{3 - 4i} = \frac{5 + 2i}{3 - 4i} \cdot \frac{3 + 4i}{3 + 4i}$$
$$= \frac{15 + 26i + 8i^2}{9 - 16i^2}$$
$$= \frac{15 + 26i - 8}{9 + 16} = \frac{7}{25} + \frac{26}{25}i.$$

If we think of $3 - 4i = 3 - 4\sqrt{-1}$, the preceding operation, where the numerator and denominator were multiplied by $3 + 4\sqrt{-1}$, is analogous to the procedure used in rationalizing denominators on page 43.

EXERCISE 49

Express in terms of i and simplify.

1. $\sqrt{-25}$. 2. $\sqrt{-98}$. 3. $\sqrt{-64x^2}$. 4. $\sqrt{-\frac{9}{25}}$.

State the two square roots of each quantity.

5. -81. 6. -49. 7. -75. 8. $-36a^6$. 9. $-\frac{3}{16}x^2$.

Perform any indicated operation and simplify to the form $a + bi$.

10. i^7. **11.** i^6. **12.** i^3. **13.** i^{15}. **14.** i^9. **15.** $3i(5i^2)$.

16. $(2i)^3$. **17.** $(3i)^5$. **18.** $\sqrt{-5}\sqrt{-20}$. **19.** $(2 + 5i)(3 - 4i)$.

20. $(3 - 2i)(-2 + 3i)$. **21.** $(\sqrt{-5} + 2)(\sqrt{-5} - 2)$.

State the conjugate of each number.

22. $5 + 3i$. **23.** $-4 - 2i$. **24.** $7i$. **25.** 6. **26.** $5 - 3\sqrt{-5}$.

Find x and y by use of Definition I, *page* 121.

27. $x + yi = 2 - 3i$. **28.** $3i = x - yi$. **29.** $2x + 3yi = 8$.

30. $x + 2 + (y - 3)i = 0$. **31.** $3x - 6 - (5 - 2y)i = 0$.

Reduce to the form $a + bi$.

32. $\dfrac{2 + 3i}{5 + 4i}$. **34.** $\dfrac{-5i + 1}{3i - 4}$. **36.** $\dfrac{3i + 4}{-i + 2}$. **38.** $\dfrac{2 + \sqrt{-4}}{3 - \sqrt{-9}}$.

33. $\dfrac{3 - 5i}{4 + 3i}$. **35.** $\dfrac{6 - i}{1 + i}$. **37.** $\dfrac{4i}{3 - 2i}$. **39.** $\dfrac{5 - \sqrt{-16}}{2 + 3i}$.

40. $\dfrac{5\sqrt{-9}}{3 - i}$. **41.** $\dfrac{5}{2i}$. **42.** $\dfrac{-4}{3i}$. **43.** $\dfrac{1}{i}$. **44.** $\dfrac{1}{i^3}$.

45. $(3\sqrt{-3})^4$. **46.** $(2i - 3)^2$. **47.** $(2i + 1)^3$. **48.** $(1 - 2i)^{-1}$.

Express the reciprocal of the number in the form $a + bi$.

49. $3 - 5i$. **50.** $3i$. **51.** $-2i$. **52.** $\sqrt{-12} + \sqrt{-75}$.

53. If $a^2 + b^2 = 1$, find the reciprocal of $a + bi$.

54. Form a quadratic equation with the roots $(a \pm bi)$.

120. Geometrical representation of complex numbers. In Figure 14, we shall represent any complex number $a + bi$ graphically by the point whose abscissa is a and ordinate is b.

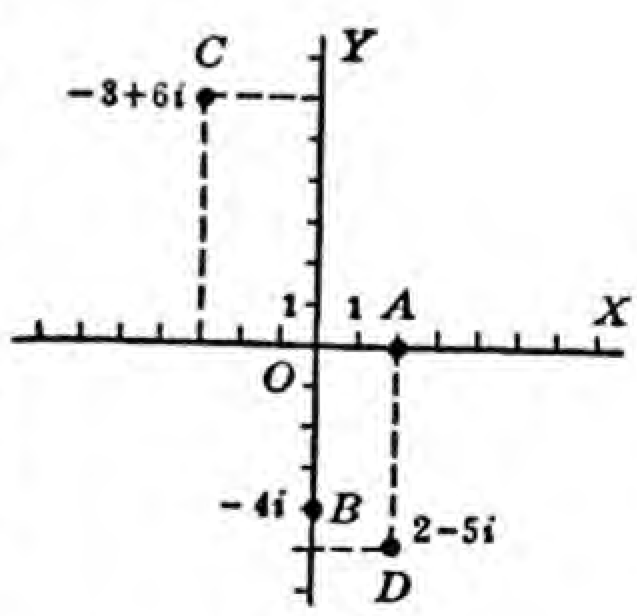


FIG. 14

ILLUSTRATION 1. In Figure 14, D represents $(2 - 5i)$. C represents $(-3 + 6i)$. The real number 2, thought of as $(2 + 0i)$, is represented by A. The pure imaginary number $-4i$, thought of as $(0 - 4i)$, is represented by B.

The imaginary part of a real number a, or $a + 0i$, is $0i$, and therefore all real numbers are represented geometrically in Figure 14 by the points on the horizontal axis OX. Similarly, all pure imaginary

numbers are represented by the points on OY, because the real part of a pure imaginary number is zero. When we use a coordinate plane for the representation of complex numbers, we refer to the horizontal axis as the *axis of real numbers*, to the vertical axis as the *axis of imaginary numbers*, and to the whole plane as the **complex plane.**

121. Geometrical construction of a sum. Let points M and N in the complex plane represent $a + bi$ and $c + di$, respectively. Connect the origin O with M and N. Complete the parallelogram with OM and ON as sides. Then, the fourth vertex, P, represents $[(a + bi) + (c + di)]$.

Proof. 1. In Figure 15,

$$OR = OS + SR = OS + MH.$$

2. Since triangles OKN and MHP are equal, $MH = OK$. Hence,

$$OR = OS + OK = a + c.$$

Similarly,

$$RP = SM + KN = b + d.$$

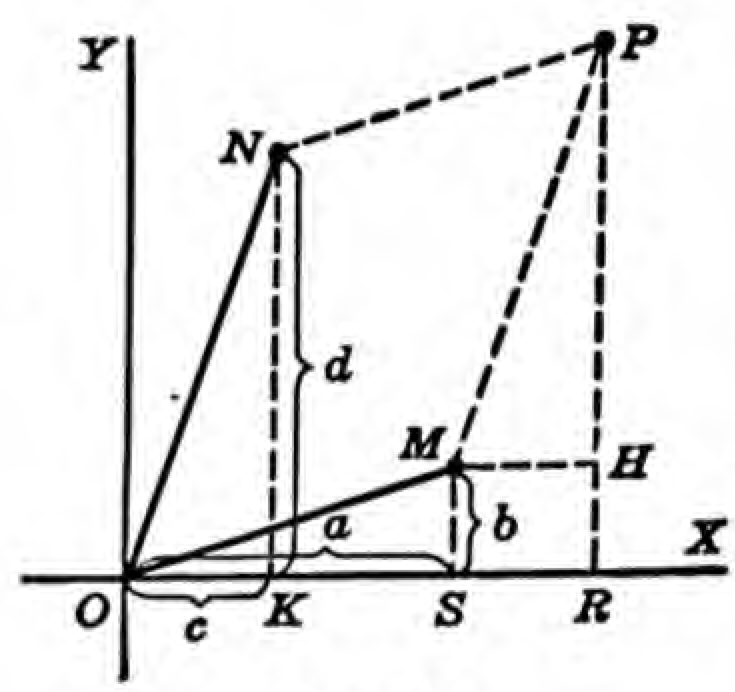


FIG. 15

3. Hence, the abscissa, OR, of P is $a + c$ and the ordinate, RP, of P is $b + d$; that is, P represents

$$[(a + c) + (b + d)i], \quad \text{or} \quad [(a + bi) + (c + di)].$$

Note 1. To *subtract* $(c + di)$ from $(a + bi)$ geometrically, we geometrically *add* $(- c - di)$ to $(a + bi)$.

ILLUSTRATION 1. In Figure 16, we find

$$[(5 + 3i) - (3 - 2i)]$$

geometrically by adding $(5 + 3i)$ and $(- 3 + 2i)$. The result, $2 + 5i$, is represented by P.

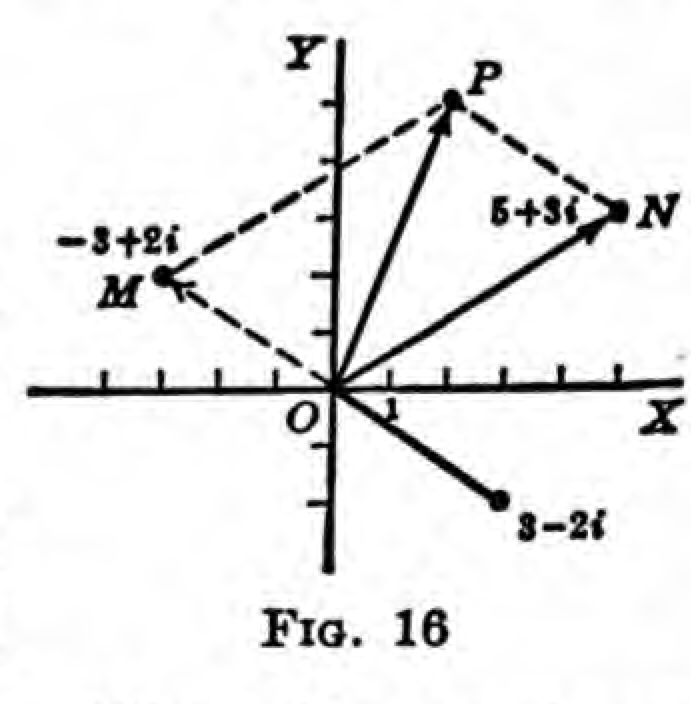


FIG. 16

In Figure 16, think of the *vectors*, OM, ON, and OP, radiating from the origin, instead of merely the *tips* of these arrows, as representing $(- 3 + 2i)$, $(5 + 3i)$, and $(2 + 5i)$, respectively. Then, the construction for obtaining P is identical with the addition of vectors OM and ON according to the **parallelogram law** as met in physics, when we combine forces, or velocities, etc.

Note 2. In the early development of algebra, the square roots of negative numbers were carefully avoided by mathematicians. Imaginary numbers were introduced in the 16th century by CARDAN but were not thoroughly appreciated until 100 years later. The words *real* and *imaginary*, as now employed in references to numbers, were introduced by DESCARTES (1637), and the symbol i for $\sqrt{-1}$ by EULER (1748). A Norwegian surveyor, WESSEL (1797), was the first to employ the geometrical representation of complex numbers on a plane. This graphical representation is of prime importance in advanced mathematics.

EXERCISE 50

Represent each complex number geometrically.

1. $5 + 2i$.
2. $7i - 5$.
3. $-5 - 2i$.
4. $i\sqrt{2} - 2$.
5. $\sqrt{-9} - 2$.
6. $8i$.
7. 6.
8. $-5i$.
9. $\sqrt{-16}$.
10. $\sqrt{-20}$.

On one plane, plot the number, its conjugate, and its negative.

11. $2 + 3i$.
12. $-2 + 4i$.
13. 7.
14. $-3i$.
15. $2 - \sqrt{-36}$.

Find each sum geometrically, and read the result from the figure.

16. $(2 + i) + (1 + 3i)$.
17. $(3 + 2i) + (2 + 3i)$.
18. $(-2 - 4i) + (-3 - 2i)$.
19. $(3 - i) + (2 + 3i)$.
20. $(3i - 2) + (-4 - 3i)$.
21. $(7 + 0i) + (0 + 3i)$.
22. $(-3i) + (-5)$.
23. $(12i) + (-3)$.
24. $(-2i) + (4)$.
25. $(-5 - 2i) + (4 + 3i) + (2 + i)$.

26. Given the point in the complex plane which represents a certain complex number N, state and demonstrate the correctness of a geometrical construction for finding (1) the negative of N; (2) the conjugate of N.

122. Trigonometric form of a complex number.* Hereafter, in the complex plane, let a single unit of length be used in measuring all distances. In Figure 17, let P represent $a + bi$, let r be the length of OP, and let $\theta = \angle XOP$. Then,

$$r = \sqrt{a^2 + b^2}; \ \tan\theta = \frac{b}{a}; \qquad (1)$$

$$a = r\cos\theta; \ b = r\sin\theta; \qquad (2)$$

$$a + bi = r(\cos\theta + i\sin\theta). \qquad (3)$$

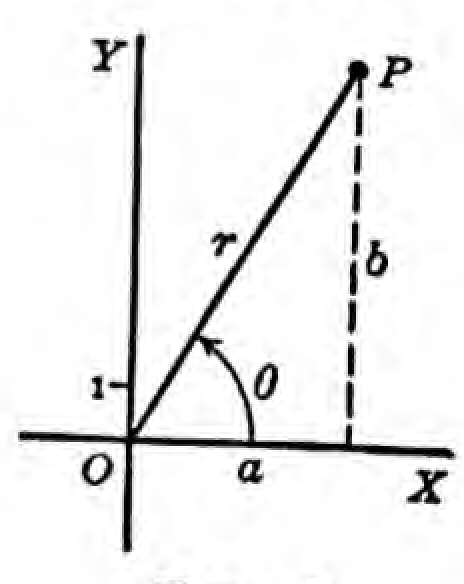


FIG. 17

* A knowledge of trigonometry is needed in the rest of this chapter. Also, the functions of 0°, 30°, 45°, 60°, 90°, etc., should be reviewed. For other angles, three-place values of the trigonometric functions are given in Table IV.

We call $r(\cos\theta + i\sin\theta)$ the **trigonometric** (or **polar**) form, θ the **amplitude** (or **argument**), and the positive length r the **absolute value** (or **modulus**) of $a + bi$. The amplitude may be taken as any angle with initial side OX and terminal side OP, because all such angles have the same trigonometric functions. Hence, if θ is one amplitude, the other permissible amplitudes are $(\theta + k\cdot 360°)$, where k is any integer. Usually, we select the amplitude as an angle which is positive or 0° and less than 360°.

ILLUSTRATION 1. In Figure 17, P represents $7(\cos 60° + i\sin 60°)$. Instead of 60°, we could use 420°, or $-$ 300°, etc. as the amplitude.

Two complex numbers are *equal* if and only if *their moduli are equal* and *their amplitudes differ at most by an integral multiple of* 360°.

To plot $r(\cos\theta + i\sin\theta)$, *construct $\angle XOP = \theta$, with $OP = r$; then P represents the given complex number.*

The number zero may be written $0 = 0\cdot(\cos\theta + i\sin\theta)$ where θ has any value. That is, the modulus of zero is 0, and the amplitude is *any* angle whatever.

To change from the trigonometric form to the form $a + bi$, we recall the values of $\sin\theta$ and $\cos\theta$ if the amplitude θ is a convenient angle, 0°, 30°, 45°, etc. Otherwise, we use Table IV.

ILLUSTRATION 2.
$$3(\cos 45° + i\sin 45°) = \tfrac{3}{2}\sqrt{2} + \tfrac{3}{2}i\sqrt{2}.$$
$$6(\cos 35° + i\sin 35°) = 6(.819 + .574i) \quad \text{(Table IV)}$$
$$= 4.914 + 3.444i.$$

If $(a + bi)$ is *real* or *pure imaginary*, we find its amplitude and absolute value by mere *inspection*.

ILLUSTRATION 3. The real number $(-4 + 0i)$, or -4, is represented by S in Figure 18. The absolute value of $(-4 + 0i)$ is OS, or 4, and the amplitude is $\angle XOS$, or 180°:
$$-4 = -4 + 0i = 4(\cos 180° + i\sin 180°),$$
which can be checked by substituting
$$\cos 180° = -1 \quad \text{and} \quad \sin 180° = 0.$$

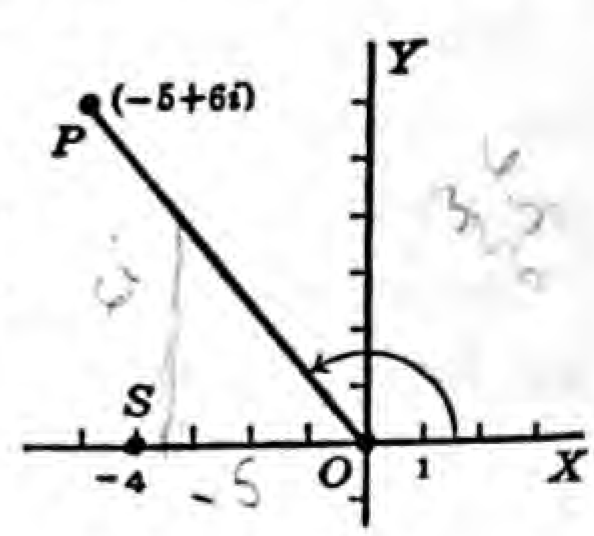


FIG. 18

The absolute value of a real number a, or $(a + 0i)$, as defined for a complex number, is $\sqrt{a^2 + 0^2}$ or $\sqrt{a^2}$, which is $+a$ if $a > 0$ and is $-a$ if $a < 0$; this is identical with $|a|$ as defined on page 3. Hence, *the two uses of the absolute value terminology are consistent.*

Also, it is consistent to use the symbol $| a + bi |$ to represent the absolute value of any complex number: $| a + bi | = \sqrt{a^2 + b^2}$.

ILLUSTRATION 4. $| -4 + 0i | = | -4 | = 4$. $| 3 + 4i | = \sqrt{25} = 5$.

123. To express $a + bi$ in the form $r(\cos \theta + i \sin \theta)$.

1. *Plot $(a + bi)$ as a point P in the complex plane, draw the modulus OP, and indicate the amplitude θ with a curved arrow.*

2. *Find r from $r = \sqrt{a^2 + b^2}$.*

3. *Find θ by noticing what quadrant it is in and using that one of the following functions which appears most convenient:*

$$\sin \theta = \frac{b}{r}; \quad \cos \theta = \frac{a}{r}; \quad \tan \theta = \frac{b}{a}.$$

EXAMPLE 1. Find the trigonometric form of $(-5 + 6i)$.

SOLUTION. 1. $r = \sqrt{61}$ and θ is in quadrant II (Figure 18).

2. $\tan \theta = -\frac{6}{5} = -1.200$. From Table IV, we find an acute angle α such that $\tan \alpha = 1.200$; we obtain $\alpha = 50.2°$. Hence,

$$\theta = 180° - 50.2° = 129.8°;$$
$$-5 + 6i = \sqrt{61}\,(\cos 129.8° + i \sin 129.8°).$$

EXERCISE 51

Plot each number. Then express it in the form $a + bi$.

1. $5(\cos 60° + i \sin 60°)$.
2. $3(\cos 210° + i \sin 210°)$.
3. $4(\cos 360° + i \sin 360°)$.
4. $5(\cos 180° + i \sin 180°)$.
5. $6(\cos 0° + i \sin 0°)$.
6. $7(\cos 270° + i \sin 270°)$.
7. $2[\cos(-45°) + i \sin(-45°)]$.
8. $3[\cos(-150°) + i \sin(-150°)]$.
9. $\cos 147° + i \sin 147°$.
10. $\cos 249° + i \sin 249°$.

Change the given number to its trigonometric form.

11. $5 + 5i$.
12. $-4 + 4i$.
13. $6 + 0i$.
14. $2i$.
15. $-3i$.
16. -5.
17. $\sqrt{3} + i$.
18. $1 - i\sqrt{3}$.
19. $-1 - i\sqrt{3}$.
20. $i - \sqrt{3}$.
21. $4 - 3i$.
22. $12 + 5i$.
23. $\cos 30° - i \sin 30°$.
24. $\cos 110° - i \sin 110°$.

Change the number and its conjugate to polar form.

25. $1 + i$. **26.** $-1 + i$. **27.** $-1 + i\sqrt{3}$. **28.** $-\sqrt{3} + i$.

29. What is the amplitude (1) of a negative real number; (2) of a pure imaginary number bi if $b < 0$?

30. Find the conjugate of $r(\cos \theta + i \sin \theta)$ in polar form.

124. Products in polar form. *The amplitude of a product of complex numbers is the sum of their amplitudes, and the absolute value of the product is the product of the absolute values of the factors.*

Proof. Consider a product of just two complex numbers:

$$r_1(\cos \theta_1 + i \sin \theta_1)\cdot r_2(\cos \theta_2 + i \sin \theta_2)$$
$$= r_1r_2(\cos \theta_1 \cos \theta_2 + i \sin \theta_1 \cos \theta_2 + i \cos \theta_1 \sin \theta_2 + i^2 \sin \theta_1 \sin \theta_2)$$
$$= r_1r_2[(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)].$$

Hence, from the addition formulas of trigonometry,

$$\boldsymbol{r_1(\cos \theta_1 + i \sin \theta_1)\cdot r_2(\cos \theta_2 + i \sin \theta_2)}$$
$$\boldsymbol{= r_1r_2[\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)].} \tag{1}$$

Illustration 1. $3(\cos 40° + i \sin 40°) \cdot 5(\cos 170° + i \sin 170°)$
$= 15(\cos 210° + i \sin 210°).$

Note 1. We extend (1) to a product of any number of factors by successive applications of (1). Thus, we use (1) twice below:

$$r_1(\cos \theta_1 + i \sin \theta_1) \cdot r_2(\cos \theta_2 + i \sin \theta_2) \cdot r_3(\cos \theta_3 + i \sin \theta_3)$$
$$= r_1r_2[\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)] \cdot r_3(\cos \theta_3 + i \sin \theta_3)$$
$$= r_1r_2r_3[\cos (\theta_1 + \theta_2 + \theta_3) + i \sin (\theta_1 + \theta_2 + \theta_3)]. \tag{2}$$

125. De Moivre's Theorem. *If n is any positive integer,*

$$\boldsymbol{[r(\cos \theta + i \sin \theta)]^n = r^n(\cos n\theta + i \sin n\theta).} \tag{1}$$

Illustration 1. From (2) in Section 124 with θ_1, θ_2, and θ_3 replaced by θ, and r_1, r_2, and r_3 replaced by r,

$$[r(\cos \theta + i \sin \theta)]^3$$
$$= r(\cos \theta + i \sin \theta) \cdot r(\cos \theta + i \sin \theta) \cdot r(\cos \theta + i \sin \theta)$$
$$= r \cdot r \cdot r \cdot [\cos (\theta + \theta + \theta) + i \sin (\theta + \theta + \theta)]$$
$$= r^3(\cos 3\theta + i \sin 3\theta).$$

Proof of formula 1. $[r(\cos \theta + i \sin \theta)]^n$ indicates the product of n factors $r(\cos \theta + i \sin \theta)$. Hence, by Section 124, the absolute value of the nth power is the product of n factors r, or r^n, and the amplitude is the sum of n amplitudes θ, or $n\theta$. Hence, (1) is true.

Example 1. Find $(1 - i)^4$ by use of De Moivre's Theorem.

Solution. 1. Express $(1 - i)$ in polar form:

$r = \sqrt{2}$; $\tan \theta = -1$, with θ in quadrant IV, so that $\theta = 315°$.

2. Hence, $\qquad (1 - i)^4 = [\sqrt{2}(\cos 315° + i \sin 315°)]^4$

$= (\sqrt{2})^4 (\cos 1260° + i \sin 1260°) = 4(\cos 180° + i \sin 180°) = -4.$

In the preceding line we noticed that $1260° = 3 \cdot 360° + 180°$.

126. Division of complex numbers in polar form. *The modulus of the quotient of two complex numbers is the quotient of their moduli, and the amplitude of the quotient is the amplitude of the dividend minus the amplitude of the divisor.*

Proof. 1. We consider $r(\cos\alpha + i\sin\alpha)/s(\cos\beta + i\sin\beta)$. On multiplying both numerator and denominator by $(\cos\beta - i\sin\beta)$, we obtain

$$\frac{r(\cos\alpha + i\sin\alpha)}{s(\cos\beta + i\sin\beta)} = \frac{r}{s}\cdot\frac{(\cos\alpha + i\sin\alpha)(\cos\beta - i\sin\beta)}{(\cos\beta + i\sin\beta)(\cos\beta - i\sin\beta)}$$

$$= \frac{r}{s}\cdot\frac{(\cos\alpha + i\sin\alpha)[\cos(-\beta) + i\sin(-\beta)]}{\cos^2\beta + \sin^2\beta}, \qquad (1)$$

because $\cos(-\beta) = \cos\beta$ and $\sin(-\beta) = -\sin\beta$.

2. In (1), we notice that $\sin^2\beta + \cos^2\beta = 1$, and apply Section 124 to the numerator:

$$\frac{r(\cos\alpha + i\sin\alpha)}{s(\cos\beta + i\sin\beta)} = \frac{r}{s}\cdot[\cos(\alpha - \beta) + i\sin(\alpha - \beta)]. \qquad (2)$$

ILLUSTRATION 1. $\dfrac{15(\cos 350° + i\sin 350°)}{5(\cos 240° + i\sin 240°)} = 3(\cos 110° + i\sin 110°).$

EXERCISE 52

Give the result in polar form, except when the final sine and cosine are known without using tables. Compute any power by use of De Moivre's Theorem.

1. $2(\cos 20° + i\sin 20°)\cdot 3(\cos 40° + i\sin 40°)$.
2. $5(\cos 30° + i\sin 30°)\cdot 2(\cos 120° + i\sin 120°)$.
3. $2(\cos 80° + i\sin 80°)\cdot 3(\cos 300° + i\sin 300°)$.
4. $3(\cos 40° + i\sin 40°)\cdot 4(\cos 350° + i\sin 350°)$.
5. $[2(\cos 20° + i\sin 20°)]^3$.
6. $[3(\cos 40° + i\sin 40°)]^5$.
7. $[2(\cos 120° + i\sin 120°)]^4$.
8. $[3(\cos 315° + i\sin 315°)]^3$.
9. $(-3 + 3i)^4$.
10. $(2 - 2i)^5$.
11. $(1 - i\sqrt{3})^5$.
12. $(i - \sqrt{3})^6$.
13. $\dfrac{10(\cos 150° + i\sin 150°)}{5(\cos 30° + i\sin 30°)}$.
14. $\dfrac{20(\cos 240° + i\sin 240°)}{5(\cos 270° + i\sin 270°)}$.
15. $\dfrac{24(\cos 160° + i\sin 160°)}{i + \sqrt{3}}$.
16. $\dfrac{8(\cos 135° + i\sin 135°)}{\sqrt{2} - i\sqrt{2}}$.
17. $\dfrac{2i\sqrt{3} - 2}{3(\cos 150° + i\sin 150°)}$.
18. $\dfrac{12(\cos 160° + i\sin 160°)}{3(\cos 20° - i\sin 20°)}$.
19. Prove that $\dfrac{1}{r(\cos\theta + i\sin\theta)} = r^{-1}[\cos(-\theta) + i\sin(-\theta)]$.

★**20.** If $z = r(\cos\theta + i\sin\theta)$ and if n is a positive integer, prove that $z^{-n} = r^{-n}[\cos(-n\theta) + i\sin(-n\theta)]$, so that *De Moivre's Theorem holds if the exponent is a negative integer.*

★**21.** On the complex plane with the origin at O, let U be the unit point on the real axis and let P and Q represent $r(\cos\alpha + i\sin\alpha)$ and $s(\cos\beta + i\sin\beta)$, respectively. Construct $\triangle UOP$ and $\angle QOM = \alpha$; complete $\triangle QOM$ similar to $\triangle UOP$. (Give the figure for α and β acute, for convenience.) Prove that *M represents the product of the given complex numbers.*

127. The nth roots of a complex number.

EXAMPLE 1. Find the cube roots of $8(\cos 150° + i\sin 150°)$.

SOLUTION. 1. Let $r(\cos\alpha + i\sin\alpha)$ be any cube root. Then,

$$8(\cos 150° + i\sin 150°) = [r(\cos\alpha + i\sin\alpha)]^3.$$

Or, by De Moivre's Theorem,

$$8(\cos 150° + i\sin 150°) = r^3(\cos 3\alpha + i\sin 3\alpha). \qquad (1)$$

2. If two complex numbers are equal, their absolute values are *equal* and their amplitudes *differ at most by some integral multiple of* 360°. Hence, from (1), the values of r and α which give cube roots satisfy the following equations:

$$r^3 = 8, \quad \text{or} \quad r = 2; \quad 3\alpha = 150° + k\cdot 360°, \quad \text{or}$$
$$\alpha = 50° + k\cdot 120°, \qquad (2)$$

where k is any integer. On placing $k = 0$, 1, and 2 in (2) we obtain 50°, 170°, and 290° as the values of α. These give the following cube roots:

$$2(\cos 50° + i\sin 50°);\quad 2(\cos 170° + i\sin 170°);$$
$$2(\cos 290° + i\sin 290°).$$

Comment. If $k = 3$ in (2), then $\alpha = 50° + 360°$, equivalent to the amplitude 50°. If $k = -1$, then $\alpha = 50° - 120° = -70° = 290° - 360°$, equivalent to 290°. Similarly, if k has any integral value in (2), the value found for α is equivalent to one of (50°, 170°, 290°). Hence, the roots obtained in Step 2 are the *only* cube roots.

Note 1. The cube roots found in Example 1 are represented by P, Q, and S in Figure 19. Notice that these points lie on a circle whose radius is 2, because 2 is the modulus of each of the roots. Moreover, P, Q, and S divide the circumference of this circle into *three equal parts* because the amplitudes of the roots are (50°, 170°, 290°), where adjacent angles differ by 120°.

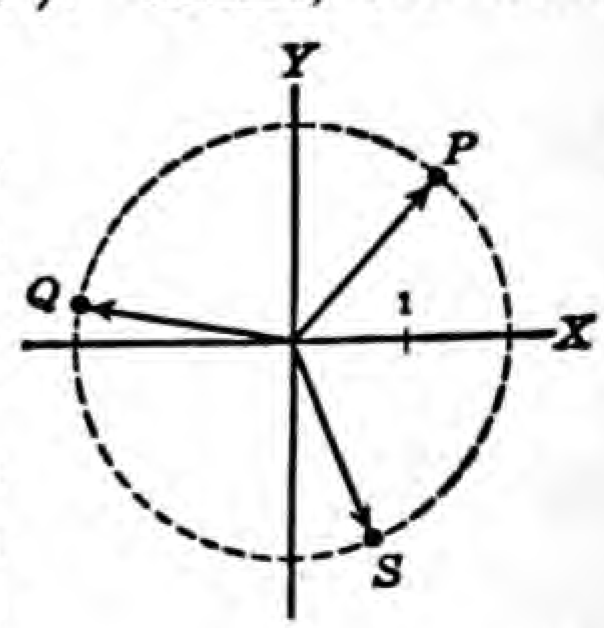


FIG. 19

THEOREM I. *If n is any positive integer and if $R > 0$, any complex number $R(\cos\theta + i\sin\theta)$ has just n distinct nth roots.*

Proof. 1. Suppose that $0 \leqq \theta < 360°$, and let $r(\cos\alpha + i\sin\alpha)$ be any nth root. Then, by De Moivre's Theorem,

$$R(\cos\theta + i\sin\theta) = [r(\cos\alpha + i\sin\alpha)]^n$$
$$= r^n(\cos n\alpha + i\sin n\alpha). \tag{3}$$

2. From (3), $r^n = R$, or $r = \sqrt[n]{R}$; and $n\alpha = \theta + k\cdot 360°$, or

$$\alpha = \frac{\theta}{n} + k\cdot\frac{360°}{n}, \tag{4}$$

where k is any integer. On placing $k = 0, 1, 2, \cdots, (n-1)$ in (4), we obtain the following n distinct values for α, all less than 360°:

$$\frac{\theta}{n};\ \left(\frac{\theta}{n} + \frac{360°}{n}\right);\ \left(\frac{\theta}{n} + 2\frac{360°}{n}\right);\ \cdots;\ \left[\frac{\theta}{n} + (n-1)\frac{360°}{n}\right]. \tag{5}$$

Corresponding to (5), we obtain the following n distinct nth roots:

$$\sqrt[n]{R}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right);\ \sqrt[n]{R}\left[\cos\left(\frac{\theta}{n} + \frac{360°}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{360°}{n}\right)\right];\ \text{etc.}$$

3. If k is given any integral value other than $0, 1, 2, \cdots, (n-1)$ in (4), we obtain a value for α differing from one of the amplitudes in (5) by an integral multiple of 360°. Hence, in (5) we have the only distinct amplitudes which give nth roots. Therefore, $R(\cos\theta + i\sin\theta)$ has *exactly* n distinct nth roots, as obtained in Step 2.

SUMMARY.* *To obtain the nth roots of $R(\cos\theta + i\sin\theta)$, place $k = 0, 1, 2, \cdots, (n-1)$ in the following formula:*

$$\sqrt[n]{R}\left[\cos\left(\frac{\theta}{n} + k\cdot\frac{360°}{n}\right) + i\sin\left(\frac{\theta}{n} + k\cdot\frac{360°}{n}\right)\right]. \tag{6}$$

ILLUSTRATION 1. To find the 5th roots of -32, or to solve $x^5 = -32$, we first write -32 in polar form:

$$-32 = 32(\cos 180° + i\sin 180°).$$

Hence, the five values of x which satisfy $x^5 = -32$ are

$$2(\cos 36° + i\sin 36°),\quad 2(\cos 108° + i\sin 108°),$$
$$2(\cos 180° + i\sin 180°),\ \text{etc.}$$

We notice that the root with amplitude 180° is -2.

* See Note 3 in the Appendix for a related extension of De Moivre's Theorem to the case of rational exponents.

Note 2. Abraham de Moivre (1667–1754) was a French mathematician who was compelled to leave France for religious reasons. He settled in London where he earned a precarious living by miscellaneous mathematical work, partly by solving problems associated with games of chance. He is particularly noted for his work entitled *The Doctrine of Chances*, which was published in 1718 and dedicated to Sir Isaac Newton.

EXERCISE 53

Leave any result in polar form, unless its amplitude is an angle whose functions are known without tables. Find the roots by the method of page 130, *except where the instructor authorizes the use of formula* 6.

1. 4th roots of $16(\cos 100° + i \sin 100°)$.

2. 5th roots of $32(\cos 150° + i \sin 150°)$.

3. Cube roots of $27(\cos 135° + i \sin 135°)$.

4. 4th roots of $81(\cos 140° + i \sin 140°)$.

5. Cube roots of 8.

6. Square roots of $-9i$.

7. Square roots of $16i$.

8. Cube roots of -1.

9. 5th roots of 32. **10.** 6th roots of 1. **11.** 4th roots of -16.

12. 4th roots of $(-8 - 8i\sqrt{3})$.

13. Cube roots of $(4\sqrt{2} - 4i\sqrt{2})$.

14. Square roots of $(24 - 7i)$.

15. 4th roots of $(15 - 20i)$.

Find all roots by the method of Section 127 *and plot them.*

16. $x^4 = 16$. **17.** $x^5 = 243$. **18.** $x^6 + 64 = 0$. **19.** $x^9 - 1 = 0$.

★**20.** Prove that, if M is any complex number and if w represents $(\cos 120° + i \sin 120°)$, one of the cube roots of 1, then the cube roots of M are $\sqrt[3]{M}$, $w\sqrt[3]{M}$, and $w^2\sqrt[3]{M}$, where $\sqrt[3]{M}$ is any cube root of M.

★**21.** If n is any positive integer, prove that the nth roots of any complex number z are equally spaced on the circumference of some circle in the complex plane.

CHAPTER TWELVE

Theory of Equations

128. Polynomials and equations of the *n*th degree. Any integral rational equation of degree n in a variable x can be put in the following form where $a_0, a_1, a_2, \cdots, a_n$ are constants and $a_0 \neq 0$:

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0. \tag{1}$$

The expression on the left side in (1) is called a *polynomial* or an **integral rational function** of degree n in x. We call (1) the *general equation of degree n.* Polynomials of degrees 1, 2, 3, and 4 are called *linear, quadratic, cubic,* and *quartic functions,* respectively. Equations of degrees 1, 2, 3, and 4 are called *linear, quadratic, cubic,* and *quartic equations,* respectively. In this chapter, any functional symbol $f(x)$, $H(x)$, etc., will represent a polynomial in x.

ILLUSTRATION 1. $3x^4 - 5x^2 = 2x - 7$ is a quartic equation.

ILLUSTRATION 2. If $f(x) = x^3 + 3x^2$, then

$$f(-2) = -8 + 12 = 4.$$

We shall restrict our illustrations and applications to polynomials and equations with *real* coefficients. However, unless otherwise stated, in any theorem and its proof, the coefficients may be any complex numbers.

129. Remainder Theorem. *If r is any constant, and if a polynomial $f(x)$ is divided by $x - r$ until a constant remainder is obtained, then this remainder equals $f(r)$.*

Proof. 1. After $f(x)$ is divided by $x - r$, let $q(x)$ represent the quotient and let R be the constant remainder. Since

$$\textit{dividend} \equiv (\textit{divisor})\cdot(\textit{quotient}) + \textit{remainder},$$

$$f(x) \equiv (x - r)q(x) + R. \tag{1}$$

2. Since (1) is true for all values of x, we may use $x = r$ in (1). Then, we obtain

$$f(r) = 0\cdot q(r) + R \quad \text{or} \quad R = f(r).$$

ILLUSTRATION 1. Let $f(x) = 5x^3 - 11x^2 - 14x - 10$. If we divide $f(x)$ by $x - 3$ by long division, we find that the constant remainder is -16. Also, by direct substitution we obtain $f(3) = 135 - 99 - 42 - 10 = -16$, which verifies the Remainder Theorem. If we divide $f(x)$ by $x + 2$, we find that the remainder is -66. Since $x + 2 = x - (-2)$, we compute $f(-2) = -66$ to verify the Remainder Theorem in this case.

130. Factor Theorem. *If $f(r) = 0$, then $x - r$ is a factor of $f(x)$. Or, if r is a root of $f(x) = 0$, then $x - r$ is a factor of $f(x)$.*

Proof. In equation 1 of Section 129, we have $R = f(r)$ and hence, by our hypothesis, $R = 0$. Therefore $f(x) = (x - r)q(x)$, and hence $x - r$ is seen to be a factor of $f(x)$.

131. Converse of the Factor Theorem. *If $x - r$ is a factor of $f(x)$, then $f(r) = 0$, or r is a root of $f(x) = 0$.*

Proof. If $f(x)$ is divided by $x - r$, the division is exact and yields an integral rational quotient $q(x)$ such that $f(x) = (x - r)q(x)$. Thus $f(r) = 0 \cdot q(r) = 0$, and hence r is a root of the equation $f(x) = 0$.

EXAMPLE 1. Is $x + 3$ a factor of $3x^3 - 2x + 5$?

SOLUTION. 1. Let $f(x) = 3x^3 - 2x + 5$, and notice that

$$x + 3 = x - (-3); \quad f(-3) = 3(-27) + 6 + 5 = -70 \neq 0.$$

2. Hence, by the preceding theorem, $x + 3$ is *not* a factor of $f(x)$.

EXERCISE 54

1. If $f(x) = 3x^2 - x + 5$, divide $f(x)$ by $x - 3$ by long division to find the constant remainder. Also, compute $f(3)$ by substitution to verify the Remainder Theorem.

2. Divide $f(x)$ of Problem 1 by $x + 2$ by long division and then compute $f(-2)$ to verify the Remainder Theorem, because $x + 2 = x - (-2)$.

Divide by long division and verify the Remainder Theorem as in Problem 2.

3. $(3x^3 + 4x^2 + 8) \div (x - 2)$.
4. $(4x^3 - 5x + 9) \div (x + 3)$.

Answer by use of the Factor Theorem and its converse; if "yes," find another factor.

5. If $f(x) = x^3 - 5x^2 + 7x - 2$, is $x - 2$ a factor of $f(x)$?
6. Is $x + 3$ a factor of $x^2 + 4x + 7$?
7. Is $x - 2$ a factor of $x^3 - 8$; of $x^3 + 8$?

Find the values of k for which $x - 3$ is a factor of $f(x)$.

8. $f(x) = 3x^3 + 2kx^2 - 5$.
9. $f(x) = k^2x^2 + 3kx - 4$.

132. Synthetic division is a process for abbreviating the division of a polynomial $f(x)$ by a binomial $x - r$.

ILLUSTRATION 1. Let us divide $5x^3 - 11x^2 - 14x - 10$ by $x - 3$.

I.

$$\begin{array}{rrrrl}
 & 5x^2 & +\ 4x & -\ 2 & = \textit{quotient}\\ \hline
5x^3 & -\ 11x^2 & -\ 14x & -\ 10 & \underline{|\,x-3}\\
\star 5x^3 & -\ 15x^2 & & & \\ \hline
 & 4x^2 & -\ 14x\star & & \\
 & \star 4x^2 & -\ 12x & & \\ \hline
 & & -\ 2x & -\ 10\star & \\
 & & \star -\ 2x & +\ 6 & \\ \hline
 & \multicolumn{3}{r}{\textit{Remainder} = -\ 16} &
\end{array}$$

II.

$$\begin{array}{|c|r|r|r|l}
\multicolumn{5}{l}{5x^2 \quad +\ 4x \quad -\ 2 = \textit{quotient}}\\ \hline
5x^3 & -\ 11x^2 & -\ 14x & -\ 10 & \underline{|\,x-3}\\
 & -\ 15x^2 & -\ 12x & +\ 6 & \\ \hline
 & 4x^2 & -\ 2x & -\ 16 &
\end{array}$$

III.

$$\begin{array}{|c|r|r|r|l}
5 & -\ 11 & -\ 14 & -\ 10 & \underline{|\ 1\ |\ -3}\\
 & -\ 15 & -\ 12 & +\ 6 & \\ \hline
5 & 4 & -\ 2 & -\ 16 &
\end{array}$$

In $x - 3$, the coefficient of x is 1; hence, at each stage in the division, the coefficient of the highest power of x in the remainder is the next coefficient in the quotient. We obtain (II) by omitting each "$\star$" term in (I) and then condensing (I) into three lines. We obtain (III) from (II) by writing only the coefficient in place of each term; we introduce "5" into the third line so that all coefficients of the quotient appear in that line, and then omit writing the quotient. (III) suggests (IV), which illustrates synthetic division. In (IV) we use "$+ 3$" instead of "$- 3$" as a multiplier so that we may *add* instead of *subtract* in obtaining the third row.

IV.

$$\begin{array}{|c|r|r|r|l}
5 & -\ 11 & -\ 14 & -\ 10 & \underline{|\,+3}\\
 & +\ 15 & +\ 12 & -\ 6 & \\ \hline
5 & +\ 4 & -\ 2 & -\ 16 &
\end{array}$$

Quotient $= 5x^2 + 4x - 2$. *Remainder* $= -\ 16$.

SUMMARY. *To divide $f(x)$ by $x - r$ by synthetic division, arrange $f(x)$ in descending powers of x, supplying each missing power with a zero as its coefficient. Then, arrange the following details in three lines.*

1. *In the first line, write the coefficients $a_0, a_1, a_2, \cdots, a_n$ of $f(x)$ in this order. Write a_0 in the first place in the third line.*

2. *Multiply a_0 by r, add the product ra_0 to a_1, and write the sum in the third line; multiply this sum by r, add the product to the next coefficient, a_2, and write the sum in the third line; etc., until finally a product is added to the last coefficient of $f(x)$.*

3. *The last number in the third line is the remainder, and the other numbers in the third line are the coefficients of the powers of x in the quotient, arranged in descending powers of x.*

EXAMPLE 1. Divide $2x^4 - 12x^2 - 5$ by $x + 3$.

SOLUTION.

$$\begin{array}{c|c|c|c|c|l} 2 & 0 & -12 & 0 & -5 & \underline{\lfloor -3} \\ & -6 & +18 & -18 & +54 & \\ \hline 2 & -6 & +6 & -18 & +49 & \end{array}$$

Quotient $= 2x^3 - 6x^2 + 6x - 18$. *Remainder* $= 49$:

$$\frac{2x^4 - 12x^2 - 5}{x+3} = 2x^3 - 6x^2 + 6x - 18 + \frac{49}{x+3}. \tag{1}$$

In Example 1, by the Remainder Theorem, it follows that $+49$ is the value of $2x^4 - 12x^2 - 5$ when $x = -3$. This illustrates the following important use of synthetic division.

SUMMARY. *To find the value of a polynomial $f(x)$ when $x = r$, divide $f(x)$ by $x - r$ by synthetic division; the remainder is $f(r)$.*

EXAMPLE 2. If $f(x) = 3x^3 + 2x - 3$, find $f(-2)$.

SOLUTION. Divide by $x - (-2)$, or $x + 2$:

$$\begin{array}{c|c|c|c|l} 3 & 0 & 2 & -3 & \underline{\lfloor -2} \\ & -6 & 12 & -28 & \\ \hline 3 & -6 & 14 & -31 & \end{array}$$

Remainder $= -31$. Hence, $f(-2) = -31$.

CHECK. By substitution, $f(-2) = -24 - 4 - 3 = -31$.

EXERCISE 55

Divide by long division and also by synthetic division. Check the remainder by finding a value of the dividend by direct substitution of a value for x.

1. $(2x^3 + 2x + 5) \div (x - 3)$.
2. $(3x^3 + 4x^2 - 5) \div (x + 2)$.

By synthetic division, find the quotient and the remainder and summarize as in equation 1, Section 132.

3. $(3x^2 + 5 - x) \div (x - 3)$.
4. $(7 + 4x^2 - 3x) \div (x + 4)$.
5. $(2x^3 + 3x^2 - 2x - 5) \div (x - 2)$.
6. $(-2x^3 + 3x^2 + 2x - 5) \div (x - 3)$.
7. $(3x^3 + 4x^2 + 8) \div (x + 2)$.
8. $(-4x^3 + 5x + 91) \div (x + 3)$.
9. $(2x^3 + 3x^2 - 2x + 3) \div (x + \frac{1}{2})$.

Solve by use of synthetic division.

10. If $f(x) = x^3 - 3x^2 + 2x - 1$, find $f(2)$; $f(-2)$.
11. If $f(x) = 2x^3 - 3x^2 - 5x + 8$, find $f(4)$; $f(-5)$.
12. If $f(x) = -5x^4 + 3x^3 + 2x - 12$, find $f(3)$; $f(-1)$.

13. If $g(z) = 4z^3 - 5z + 9$, find $g(\frac{1}{2})$; $g(-3)$.

14. Find $(x^4 - 4x^3 + 13x^2 - 36x + 36) \div (x - 2)^2$, by dividing twice in succession by $x - 2$.

15. Prove that $x - 1$ is a factor of $x^7 - 1$ by use of the factor theorem, and find another factor by synthetic division.

16. Prove that $x - c$ is a factor of $x^5 - c^5$ and find another factor by synthetic division.

★17. By use of the Factor Theorem or its converse, prove items I to IV of Section 26, page 16. In (I), (II), and (III), find another factor by synthetic division.

133. Graphs of polynomials. In graphing a polynomial $f(x)$, synthetic division may be used in computing values of $f(x)$.

ILLUSTRATION 1. A graph of the function $f(x) = x^3 - 12x + 3$ is given in Figure 20. The following values of $f(x)$ were computed and the graph was drawn through the corresponding points.

WHEN $x =$	-4	-3	-2	-1	0	1	2	3	4
THEN $f(x) =$	-13	$+12$	$+19$	$+14$	$+3$	-8	-13	-6	$+19$

On the graph, point M (where $x = -2$) is *higher* than any neighboring point of the curve. We call M a **maximum** point; we say that $f(x)$ has a *maximum value* when $x = -2$ because $f(-2)$ is greater than any other value of $f(x)$ if *x is sufficiently near to $x = -2$*. Point m (where $x = 2$) is *lower* than any neighboring point of the graph. We call m a **minimum** point and say that $f(x)$ has a *minimum value* when $x = 2$; $f(2)$ is less than $f(x)$ if x has any value *sufficiently near to* $x = 2$.

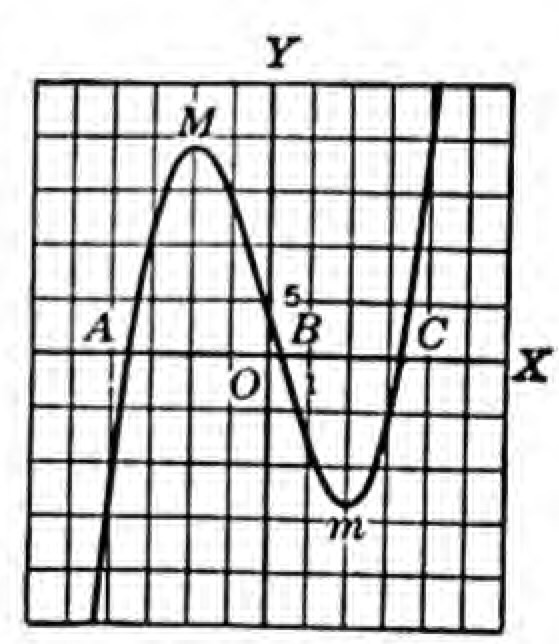


FIG. 20

Note 1. Figure 20 and curves I, II, and III on page 138 illustrate the different types met as the graphs of cubic functions. Curve IV in Figure 21 is the graph of a certain quartic function. Notice that curve IV has two minima and one maximum.

In more advanced mathematics it is proved that the graph of a polynomial $f(x)$ of degree n is *a continuous curve, with at most $(n - 1)$ maxima and minima, and with no sharp corners.* A polynomial $f(x)$ is called a **continuous, single-valued function** of x, because the graph of $f(x)$ is a *continuous* curve, and because, to each value of x, there corresponds a *single* value of $f(x)$.

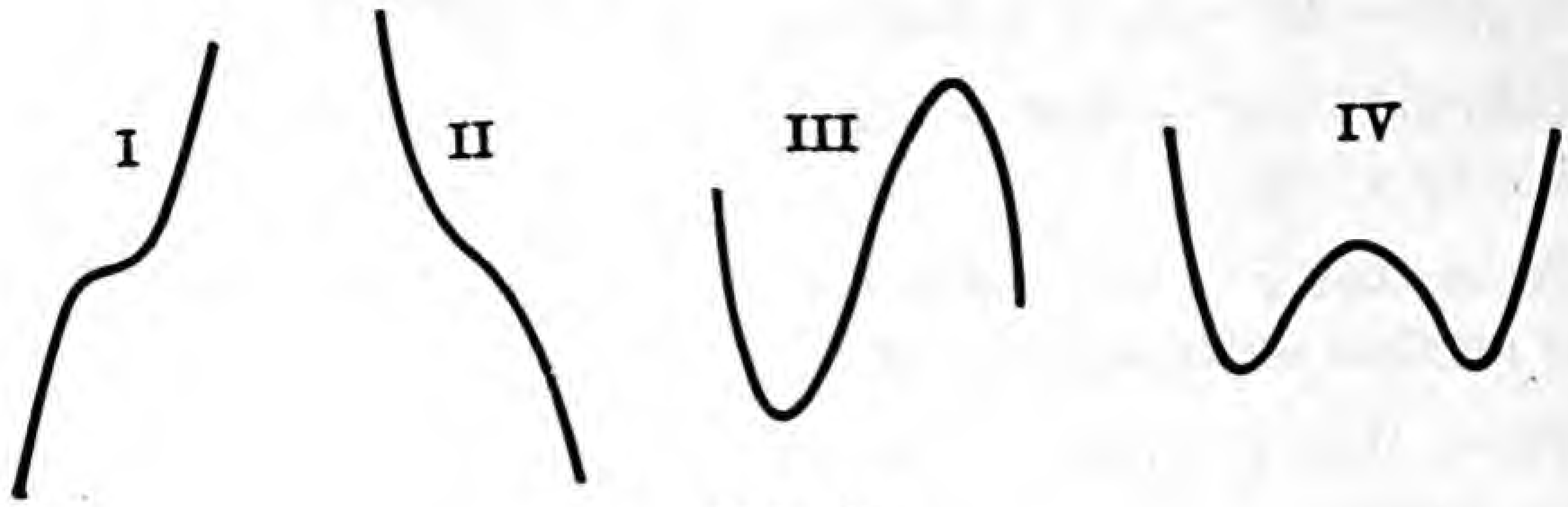


FIG. 21

134. Graphical solution of equations. Recall that the real roots of $f(x) = 0$ are the abscissas of the points where the graph of $f(x)$ meets the x-axis.

EXAMPLE 1. Solve graphically: $x^3 + 3 = 12x.$

SOLUTION. 1. Transpose $12x$: $x^3 - 12x + 3 = 0.$

2. Let $f(x) = x^3 - 12x + 3$. A graph of $f(x)$ is seen in Figure 20, page 137; at the points A, B, and C the value of $f(x)$ is zero. Hence, the abscissas of A, B, and C are the real roots of $f(x) = 0$. These roots are, approximately, $x = -3.6$; $x = .3$; $x = 3.3$.

Note 1. In a later section, we shall refine the graphical method so that it will yield the real roots with any specified degree of accuracy. At present, we shall be content to obtain, graphically, only rough approximations to the real roots.

EXERCISE 56

Graph each polynomial.

1. x^3.
2. $-x^3$.
3. x^4.
4. $-x^4$.
5. $x^3 + 2x^2 - 4x + 3$.
6. $-2x^3 - 2x^2 + 2x - 3$.
7. $x^3 - 2x^2 + 2x - 3$.
8. $-x^3 - x^2 - x$.
9. $-3x^4 - 4x^3 + 12x^2 + 6$.
10. $3x^4 - 8x^3 - 6x^2 + 24x - 10$.

Obtain approximate values of the real roots graphically.

11. $x^2 - 3x - 5 = 0$.
12. $x^3 + x^2 - 6x - 7 = 0$.
13. $x^3 - 4x^2 - 2x + 8 = 0$.
14. $2x^4 - 7x^2 + 2 = 0$.
15. $x^3 + 6x^2 + 12x - 2 = 0$.

135. Fundamental theorem of algebra. *Every integral rational equation in a single variable x has at least one root.*

Note 1. The preceding theorem was first proved in 1799 by the great German mathematician JOHANN KARL FRIEDRICH GAUSS (1777–1855). The proof of this theorem is beyond the scope of this book.

THEOREM I. *If $f(x)$ is any polynomial of degree n in x, then there exist n linear factors whose product is $f(x)$.*

Proof. 1. Suppose that $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$.

2. By the fundamental theorem, the equation $f(x) = 0$ has at least one root. Let r_1 be this root; then, by the factor theorem, $f(x)$ has $x - r_1$ as a factor. If we let $Q_1(x) = [f(x) \div (x - r_1)]$, then $Q_1(x)$ is a polynomial of degree $n - 1$ whose term of highest degree is a_0x^{n-1}:

$$f(x) = (x - r_1)Q_1(x). \tag{1}$$

3. By the fundamental theorem, $Q_1(x) = 0$ has a root, r_2. Therefore, $x - r_2$ is a factor of $Q_1(x)$, and $Q_1(x) = (x - r_2)Q_2(x)$, where $Q_2(x)$ is a polynomial of degree $n - 2$ whose term of highest degree is a_0x^{n-2}. On using the expression just found for $Q_1(x)$, from (1) we obtain $f(x) = (x - r_1)(x - r_2)Q_2(x)$.

4. On continuing this process through n steps, we obtain n numbers $r_1, r_2, \cdots, r_n$ and a function $Q_n(x)$ such that

$$f(x) = (x - r_1)(x - r_2)\cdots(x - r_n)Q_n(x), \tag{2}$$

where a_0 is the coefficient of the term of highest degree in $Q_n(x)$. Moreover, the degree of $Q_n(x)$ is $(n - n)$ or zero. That is, $Q_n(x)$ is a constant and hence $Q_n(x) = a_0$. Therefore,

$$f(x) = a_0(x - r_1)(x - r_2)\cdots(x - r_n). \tag{3}$$

Note 2. For a given function $f(x)$, all of the numbers $r_1, r_2, \cdots, r_n$ may not be distinct, and some or all may be imaginary.

THEOREM II. *Every equation $f(x) = 0$ of degree n has at most n distinct roots.*

Proof. 1. From Theorem I, $f(x) = a_0(x - r_1)(x - r_2)\cdots(x - r_n)$. Hence, by the converse of the factor theorem, each of $r_1, r_2, \cdots, r_n$ is a root of $f(x) = 0$. We notice that all of these roots may not be distinct.

2. If r is any number different from all of $r_1, r_2, \cdots, r_n$, then

$$f(r) = a_0(r - r_1)(r - r_2)\cdots(r - r_n) \neq 0,$$

because no factor of $f(r)$ is zero. Therefore, r is not a root of $f(x) = 0$, and hence $f(x) = 0$ has no roots other than $r_1, r_2, \cdots, r_n$.

If a root R occurs just once among $r_1, r_2, \cdots, r_n$, then R is called **a simple root.** If R occurs exactly h times or, in other words, *if $(x - R)^h$ is the highest power of $(x - R)$ which is a factor of $f(x)$, R*

is called a **multiple root** of $f(x) = 0$, whose **multiplicity** is h. Roots of multiplicities 2 and 3 are called **double** and **triple roots,** respectively. The preceding theorem may be restated as follows:

Every equation of degree n has exactly n roots $r_1, r_2, \cdots, r_n$, where a root of multiplicity h is counted as h roots.

★COROLLARY 1. *If two polynomials*

$$a_0x^n + a_1x^{n-1} + \cdots + a_n,$$

and

$$b_0x^n + b_1x^{n-1} + \cdots + b_n,$$

each of degree not greater than n, are equal in value for more than n distinct values of x, then the polynomials are identical term by term; that is, $a_0 = b_0$, $a_1 = b_1$, $\cdots$, $a_n = b_n$, and hence the polynomials are equal for all values of x.

Proof. By assumption, the equation

$$a_0x^n + a_1x^{n-1} + \cdots + a_n - (b_0x^n + b_1x^{n-1} + \cdots + b_n) = 0, \quad (4)$$

or,

$$(a_0 - b_0)x^n + (a_1 - b_1)x^{n-1} + \cdots + (a_n - b_n) = 0, \quad (5)$$

has more than n distinct roots. If any one of the coefficients $(a_0 - b_0)$, $(a_1 - b_1)$, $\cdots$, $(a_n - b_n)$ in (5) were *not* zero, then (5) would be an equation of degree n or less, with more than n distinct roots. This fact would contradict the preceding theorem. Hence, all coefficients in (5) must be zero; that is, $a_0 = b_0$, $a_1 = b_1$, $\cdots$, $a_n = b_n$.

EXAMPLE 1. Form an equation with the following roots and no others: -2, 4 as a triple root, $(3 + i\sqrt{2})$, and $(3 - i\sqrt{2})$.

SOLUTION. By use of equation 3 with $a_0 = 1$, one equation is

$$(x + 2)(x - 4)^3[x - (3 + i\sqrt{2})][x - (3 - i\sqrt{2})] = 0,$$

or,

$$(x + 2)(x - 4)^3(x^2 - 6x + 11) = 0.$$

EXAMPLE 2. Solve: $x(x + 5)(x^2 + 4x + 7) = 0$.

SOLUTION. 1. If $x + 5 = 0$, then $x = -5$; another root is $x = 0$.

2. If $x^2 + 4x + 7 = 0$, then $x = \frac{1}{2}(-4 \pm \sqrt{-12}) = -2 \pm i\sqrt{3}$.

136. Theorem on imaginary roots. *If an imaginary number $(a + bi)$ is a root of an equation $f(x) = 0$ with real coefficients, then the conjugate imaginary $(a - bi)$ also is a root; that is, imaginary roots occur in pairs.*

ILLUSTRATION 1. If $f(x)$ has real coefficients and if $(3 + 2i)$ is a root of $f(x) = 0$, then $(3 - 2i)$ also is a root.

Note 1. A proof of the preceding theorem is given in the Appendix, Note 4.

ILLUSTRATION 2. In Section 82, page 75, we observed that, when the roots of $ax^2 + bx + c = 0$ are imaginary, these roots are *conjugate* imaginary numbers. This is a special case of the preceding theorem.

COROLLARY 1. *Every polynomial $f(x)$ with real coefficients can be expressed as a product of real linear and quadratic factors.*

Proof. 1. If $r_1, r_2, \cdots, r_n$ are the roots of $f(x) = 0$, then

$$f(x) = a_0(x - r_1)(x - r_2)\cdots(x - r_n). \tag{1}$$

2. By the preceding theorem, if $f(x)$ has an imaginary factor $[x - (a + bi)]$, then $[x - (a - bi)]$ also is a factor. Since

$$[x - (a + bi)][x - (a - bi)] = x^2 - 2ax + a^2 + b^2,$$

which has real coefficients, all imaginary factors can be combined by pairs to give real quadratic factors.

EXERCISE 57

Solve without multiplying the factors.

1. $(x - 2)(x + 5)(x + 7) = 0.$
2. $(x^2 - 3x)(x^2 - 5x + 8) = 0.$
3. $(x^2 + 4x)(3x^2 + 7x + 9) = 0.$
4. $(4x^2 + 9)(2x - 3) = 0.$

Form an equation with integral coefficients having the given numbers and no others as roots.

5. $3;\ 3;\ -5;\ -4.$
6. $3;\ -2;\ -2;\ 1.$
7. $\frac{2}{3};\ \frac{3}{4};\ -2;\ 0.$
8. $\frac{2}{5};\ \frac{1}{3};\ 5;\ 0.$
9. $\pm\sqrt{3};\ \pm 2i.$
10. $\pm 2i;\ 3;\ \pm\sqrt{2}.$
11. $\pm 4i;\ \frac{3}{5};\ \frac{2}{3}.$
12. $3;\ (1 \pm \sqrt{2}).$
13. $(2 \pm \sqrt{3});\ \pm i.$
14. $(3 \pm \frac{1}{2}\sqrt{2});\ 1.$
15. $(2 \pm i\sqrt{3});\ 2.$
16. $(\frac{1}{2} \pm i\sqrt{3});\ -1.$
17. -2 as a triple root.
18. 2 as a root of multiplicity 4.
19. 3 as a double root and $\frac{1}{3}$ and $-\frac{1}{3}$ as simple roots.

If $f(x) = 0$ is an equation with real coefficients which has the given number as a root, what other number is a root of $f(x) = 0$?

20. $(2 - 3i).$
21. $(-2 + 4i).$
22. $(-5 - 5i).$
23. $(-3 - 4i).$
24. Form a cubic equation with $(2 + i)$ and 3 as roots.
25. Form a quartic equation with $(3 - 2i)$ and $(1 + i)$ as roots.
26. Prove that a cubic equation, with real coefficients, has either three real roots, or one real and two imaginary roots.

State and prove theorems similar to those of Problem 26.

27. For an equation of degree 4.
28. For an equation of degree 5.

Prove the fact and find all roots by use of repeated synthetic division.

29. $x^4 - 4x^3 + 13x^2 - 36x + 36 = 0$ has 2 as a double root.

30. $x^4 + 4x^3 - 6x^2 - 36x - 27 = 0$ has -3 as a double root.

31. Prove that every equation of odd degree with real coefficients has at least one real root.

★**32.** Prove that, if an equation with rational coefficients has a root $(a + \sqrt{b})$ where a and b are rational but $\sqrt{b}$ is irrational, then the equation also has the root $(a - \sqrt{b})$. Use the method of Note 4 of the Appendix.

★**137. Graph of a factored polynomial.** Suppose that r is real and that $(x - r)^h$ is the *highest power* of $x - r$ which is a factor of $f(x)$. Or, in other words, suppose that r is a root of $f(x) = 0$ of multiplicity h. Then, the following statements I and II summarize certain facts about the graph of $f(x)$ which are established in more advanced courses in mathematics.

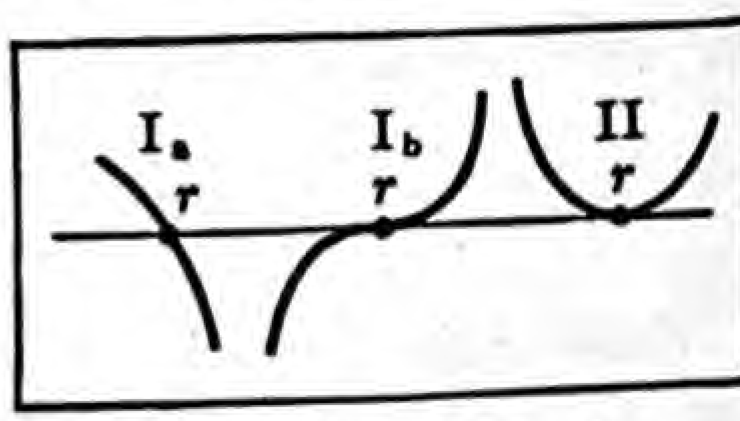


FIG. 22

I. *If h is an odd integer, the graph of $f(x)$ crosses the x-axis at $x = r$. If $h = 1$, the graph cuts the x-axis sharply at $x = r$* (see $\mathrm{I_a}$, Figure 22). *If h is an odd integer greater than 1, the graph on each side of $x = r$ is tangent to the x-axis at $x = r$* (see $\mathrm{I_b}$, Figure 22).

II. *If h is an even integer, the graph of $f(x)$ is entirely on one side of the x-axis, near $x = r$, and is tangent to the x-axis at $x = r$* (see II, Figure 22).

If we know the factors of $f(x)$, facts I and II simplify the construction of the graph of $f(x)$ around points where $f(x) = 0$.

EXAMPLE 1. Graph the function $\quad f(x) = (x + 4)^3(x + 1)(x - 1).$

SOLUTION. 1. The graph meets the x-axis where $x = -4$, and $x = \pm 1$, which are the roots of $f(x) = 0$.

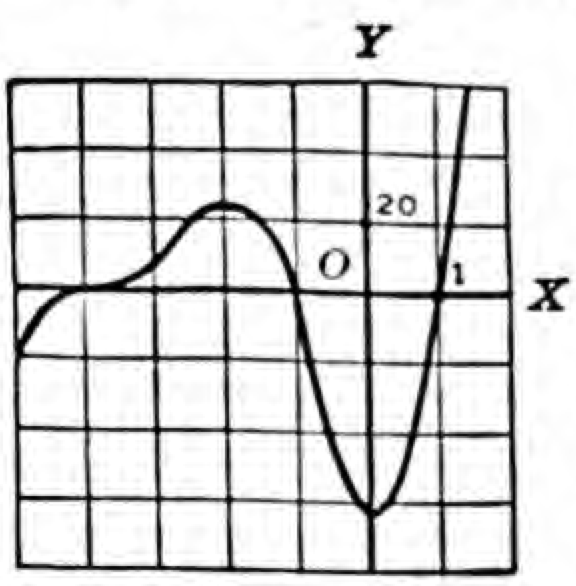


FIG. 23

2. If x is negative and has a very large numerical value, -20 for instance, then $f(x)$ is negative, and its graph is *below* the x-axis. Since $(x + 4)^3$ is an odd power, the graph near $x = -4$ on the x-axis is like $\mathrm{I_b}$ in Figure 22. At $x = \pm 1$ on the x-axis, we make the graph cut the axis sharply. In addition to these observations, we make use of a few points not on the x-axis in graphing $f(x)$ in Figure 23.

WHEN $x =$	-5	-4	-3	-2	-1	0	1	2
THEN $f(x) =$	-24	0	8	24	0	-64	0	648

Note 1. If $f(x)$ has a quadratic factor which is the simplified form of a product $[x - (a + bi)][x - (a - bi)]$, corresponding to a pair of imaginary roots of $f(x) = 0$, this quadratic factor is zero only when $x = a \pm bi$. Hence, corresponding to this factor there are no intersections of the graph of $f(x)$ with the x-axis. Thus, the graph of the function $f(x) = (x + 3)(x^2 - 2x + 5)$ would meet the x-axis only at $x = -3$ because $x^2 - 2x + 5 = 0$ if $x = 1 \pm 2i$.

★EXERCISE 58

Graph without expanding.

1. $x + 2$.
2. $(x - 3)^3$.
3. $(x - 2)^2$.
4. $(x + 1)^4$.
5. $(x - 2)(x + 3)$.
6. $(x + 2)(4 - x)$.
7. $(x - 2)(x + 3)(x - 4)$.
8. $(x + 2)(x - 3)(x + 5)$.
9. $(2 - x)(x - 1)(x + 1)$.
10. $(x - 1)^2(x + 2)$.
11. $(4 - x)(x + 1)^2$.
12. $(x - 1)^2(x - 3)^3$.
13. $x^2(x + 1)(x - 3)$.
14. $(x + 3)^2(x^2 + 2x - 3)$.
15. $(2 - x)^3(x + 3)$.

138. Relation between the roots of $f(x) = 0$ and those of $f(-x) = 0$. Let $r_1, r_2, \cdots, r_n$ be the roots of $f(x) = 0$. If we place $x = -X$ in $f(x) = 0$, we obtain $f(-X) = 0$, which is satisfied if and only if

$$-X = r_1, \quad -X = r_2, \cdots, \quad -X = r_n,$$

or,

$$X = -r_1, \quad X = -r_2, \cdots, \quad X = -r_n.$$

Hence, the roots of $f(-X) = 0$ are the *negatives* of the roots of $f(x) = 0$.

THEOREM I. *To obtain the equation $f(-X) = 0$, whose roots are the negatives of the roots of a given equation $f(x) = 0$, change the sign of the coefficient of each term of odd degree in $f(x) = 0$, and then change x to X.*

Proof. Let the given equation $f(x) = 0$ be

$$a_n + a_{n-1}x + a_{n-2}x^2 + a_{n-3}x^3 + \cdots + a_0x^n = 0. \qquad (1)$$

If we replace x by $-X$ in (1), we obtain $f(-X) = 0$, or

$$a_n + a_{n-1}(-X) + a_{n-2}(-X)^2 + a_{n-3}(-X)^3 + \cdots + a_0(-X)^n = 0.$$

Hence,

$$a_n - a_{n-1}X + a_{n-2}X^2 - a_{n-3}X^3 + \cdots \text{ etc.} = 0,$$

which agrees with Theorem I.

EXAMPLE 1. Obtain an equation whose roots are the negatives of the roots of $x^4 + 3x^3 - 3x^2 - 5x + 8 = 0$.

SOLUTION. By use of Theorem I, or by the direct substitution $x = -X$, the desired equation is found to be $X^4 - 3X^3 - 3X^2 + 5X + 8 = 0$.

EXERCISE 59

Find an equation whose roots are the negatives of those of the given equation.

1. $2x^3 - 5x^2 + 3x - 5 = 0.$
2. $3x^3 + 4x^2 - 2x - 3 = 0.$
3. $4x^4 + 3x^3 + 2x - 4 = 0.$
4. $2x^5 - 2x^4 - 3x + 7 = 0.$
5. $3x - x^3 - 5 = 0.$
6. $x^{10} - x^7 + 3x^6 = 4x^4 - 2.$
7. $-3x^5 + 2x^4 - 5x = 7.$
8. $x^6 - x^4 + x - 3 = 0.$

139. A theorem concerning the signs of the coefficients. In a polynomial $f(x)$ arranged in descending powers of x, if two successive terms differ in sign, there is said to be a **variation in sign.** In counting variations, zero terms are disregarded.

THEOREM I. *If $g(x)$ is any polynomial and if r is positive, then $(x - r)g(x)$ has at least one more variation of sign than $g(x)$.**

ILLUSTRATION 1. If $\quad g(x) = 3x^2 - 5x + 7,\quad$ then

$$(x - 2)g(x) = 3x^3 - 11x^2 + 17x - 14,$$

which shows *three* variations in sign whereas $g(x)$ shows just *two* variations.

140. Descartes' rule of signs. *The number of positive roots of $f(x) = 0$ cannot exceed the number of variations of sign in $f(x)$.*

Proof. 1. Let $r_1, r_2, \cdots, r_k$ be the positive roots of $f(x) = 0$. Then, by the factor theorem,

$$f(x) = (x - r_1)(x - r_2)\cdots(x - r_k)Q(x),$$

where
$$Q(x) = \frac{f(x)}{(x - r_1)(x - r_2)\cdots(x - r_k)}.$$

2. By Section 139, $(x - r_1)Q(x)$ has at least one more variation of sign than $Q(x)$, or at least 1 variation; $(x - r_2)(x - r_1)Q(x)$ has at least one more variation of sign than $(x - r_1)Q(x)$, or at least 2 variations. Finally, $(x - r_1)(x - r_2)\cdots(x - r_k)Q(x)$, or $f(x)$ itself, has at least k variations of sign, where k is the number of positive roots of $f(x) = 0$.

The roots of $f(-x) = 0$ are the negatives of the roots of $f(x) = 0$. Hence, the *negative* roots of $f(x) = 0$ give rise to the *positive* roots of $f(-x) = 0$. Therefore we obtain the following corollary.

COROLLARY 1. *The number of negative roots of $f(x) = 0$ cannot exceed the number of variations of sign in $f(-x)$.*

* For a proof of the theorem, see Note 5 in the Appendix.

Without actually solving an equation, we can obtain useful information about its roots by use of Descartes' rule of signs, in combination with previous theorems about the roots.

Example 1. State what can be learned about the roots of the following equation without solving it:

$$2x^4 + 5x^2 - 4x - 1 = 0.$$

Solution. 1. Let $f(x)$ represent the left member. $f(x)$ has one variation of sign; hence, there is *at most one* positive root.

2. We obtain $f(-x) = 2x^4 + 5x^2 + 4x - 1$, which has one variation of sign. Hence, $f(x) = 0$ has *at most one* negative root.

3. Since $f(x) = 0$ has four roots, *at least* two roots must be imaginary. Since imaginary roots occur in pairs, the following possibilities about the roots exist, as far as our information is concerned: (a) one positive, one negative, and two imaginary; or (b) all four imaginary.

Comment. Possibility b is eliminated by thinking of a rough graph of $f(x)$. We have $f(-3) = 218$; $f(0) = -1$; $f(3) = 194$. Hence, the graph of $f(x)$ crosses the x-axis between $x = -3$ and $x = 0$ and between $x = 0$ and $x = 3$, so that (a) in Step 3 states the true facts.

Note 1. René Descartes (1596–1650) was a French mathematician, physicist, and philosopher. He is particularly noted as the inventor of analytic geometry, a cornerstone of modern mathematics. The basic idea of analytic geometry is the notion of the graph of an equation on a coordinate system. In honor of Descartes, rectangular coordinates are sometimes called *Cartesian* coordinates.

★*Note 2.* A stronger form of Descartes' Rule of Signs than that given in this section is found in concluding remarks in Note 5 of the Appendix. The answers listed for Exercise 60 do not employ the stronger form.

EXERCISE 60

Without solving, obtain information about the roots by use of general theorems and describe the possibilities which exist.

1. $x^3 + 3x - 4 = 0.$

2. $3x^3 + 2x + 5 = 0.$

3. $x^4 + 2x^2 + 1 = 5x^3.$

4. $x^4 + 3x^2 + 7 = 0.$

5. $3x^4 + 2x^2 + 2x = 3.$

6. $x^3 - 4x^2 + 2x = 3.$

7. $2x^4 - 2x^2 - 3 = 0.$

8. $x^7 - x^4 + 3 = 0.$

9. $x^6 - 4 = 0.$

10. $x^7 - x^3 = 1 + x.$

11. $x^4 + 3x = 2.$

12. $x^5 - 1 = 0.$

13. $x^6 + 3 = 0.$

14. $x^7 + 3 = 0.$

15. $x^7 + 3 = x^2.$

Given that all roots are real, determine their signs.

16. $x^3 - 2x^2 - 5x + 6 = 0.$

17. $x^3 - 6x^2 + 11x - 6 = 0.$

18. $x^6 - 3x^4 + 3x^2 - 1 = 0.$

19. $x^5 - 2x^3 + x + 2x^4 + 2 = 4x^2.$

★*Without solving, find the number of imaginary roots which exist.*

20. $x^6 + 2x^4 + 3x^3 - 2 = 0.$

21. $x^4 + 3x^3 + x - 4 = 0.$

22. $3x^4 - 2x^3 - 2x - 6 = 0.$

23. $3x^6 - 2x^4 - 5 = 0.$

★141. Limits for the roots. If no real root of $f(x) = 0$ is greater than L, we call L an **upper limit** for the real roots of $f(x) = 0$. If no real root is *less* than l, we call l a **lower limit** for the real roots. An upper limit can be found by use of the following theorem, which we accept without proof and easily justify in each application. A lower limit for the roots can be found by* applying the theorem to $f(-x) = 0$, whose roots are the negatives of the roots of $f(x) = 0$.

THEOREM I. *If k is positive or zero, and if all numbers in the third row of the synthetic division of $f(x)$ by $x - k$ are of the same sign or zero, then no real root of $f(x) = 0$ is greater than k.*

In applying Theorem I to an equation where the coefficient of the highest power of x is *negative*, for convenience in reasoning we first multiply both sides by -1 to make this coefficient *positive*.

EXAMPLE 1. Find limits for the real roots of

$$f(x) = x^3 + 3x^2 - 12x - 9 = 0.$$

SOLUTION. 1. On dividing $f(x)$ by $x - 3$ by synthetic division, all numbers in the third line are found to be positive, and $f(3) = 9$.

1	3	-12	-9	3
	3	18	18	
1	6	6	9	

Hence, if we should divide $f(x)$ by $x - a$, where a is any number *greater* than 3, we would find that $f(a) > 9$, because each number in the second row in this new division would be greater than the corresponding number in the division by $x - 3$. Hence, if $a > 3$, then a is not a root of $f(x) = 0$; or, 3 is an *upper* limit for the real roots of $f(x) = 0$.

2. To find a lower limit, consider $f(-x) = -x^3 + 3x^2 + 12x - 9 = 0$;

or, $$x^3 - 3x^2 - 12x + 9 = 0.$$

By the method of Step 1, we find that 6 is an upper limit for the roots of $f(-x) = 0$. Hence, -6 is *less* than any root of $f(x) = 0$, or -6 is a *lower* limit for the roots.

* Or, by the following companion to Theorem I: if k is *negative*, and if the numbers in the third row of the synthetic division of $f(x)$ by $x - k$ alternate in sign, then k is a *lower limit* for the roots of $f(x) = 0$. The student may desire to use this method to verify the lower limit in Example 1.

★EXERCISE 61

Find limits for the real roots of each equation.

1. $x^3 + 2x^2 - 12x + 6 = 0.$
2. $3x^4 - 2x^3 - 18x^2 = 60.$
3. $x^3 - 3x^2 - 80 = 0.$
4. $x^3 - 2x^2 = 35x - 14.$
5. $4x^4 - 15x^3 + 12x^2 = 15.$
6. $x^5 - 4x^4 - 16x - 72 = 0.$

142. Theorem on rational roots. *If an equation*

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0, \tag{1}$$

with integral coefficients, has a rational root c/d, where c/d is in lowest terms, then c is a factor of a_n and d is a factor of a_0.

Proof. 1. By hypothesis, c and d are integers with no common factor except ± 1. On substituting c/d for x in (1), we obtain

$$a_0\frac{c^n}{d^n} + a_1\frac{c^{n-1}}{d^{n-1}} + a_2\frac{c^{n-2}}{d^{n-2}} + \cdots + a_{n-1}\frac{c}{d} + a_n = 0. \tag{2}$$

2. On multiplying both sides of (2) by d^n we find

$$a_0c^n + a_1c^{n-1}d + a_2c^{n-2}d^2 + \cdots + a_{n-1}cd^{n-1} + a_nd^n = 0. \tag{3}$$

3. On subtracting a_0c^n from both sides of (3), we obtain

$$d(a_1c^{n-1} + a_2c^{n-2}d + \cdots + a_{n-1}cd^{n-2} + a_nd^{n-1}) = -a_0c^n. \tag{4}$$

In (4), all letters represent integers and d is a factor of the left side. Hence, d must be a factor of a_0c^n. But, unless $d = \pm 1$, d is not a factor of c^n because d is not a factor of c. Hence, d is a factor of a_0.

4. On subtracting a_nd^n in (3), we obtain

$$a_0c^n + a_1c^{n-1}d + \cdots + a_{n-1}cd^{n-1} = -a_nd^n. \tag{5}$$

In (5), c is a factor of the left side and hence is a factor of a_nd^n. But, unless $c = \pm 1$, c is not a factor of d^n. Hence, c is a factor of a_n.

COROLLARY 1. *Any rational root of an equation*

$$x^n + b_1x^{n-1} + b_2x^{n-2} + \cdots + b_{n-1}x + b_n = 0, \tag{6}$$

with integral coefficients, is an integer and an exact divisor of b_n.

Proof. By the theorem, if c/d is a root of (6), then d is a factor of the coefficient of x^n, and c is a factor of b_n. Since the coefficient of x^n is 1, hence $d = \pm 1$, and therefore the rational root c/d is an integer, $\pm c$, which is a factor of b_n.

Note 1. For reference, one may say that equation 6 is in the **b-form**; its essential feature is that the coefficient of x^n is 1.

In solving an equation, whenever a rational root is found, *depress* the degree of the original equation $f(x) = 0$ by removing the factor of $f(x)$ corresponding to the known root. Then, continue the solution by finding the roots of the **depressed equation.**

EXAMPLE 1. Find all rational roots of

$$f(x) = x^4 - 6x^3 + 3x^2 + 24x - 28 = 0.$$

SOLUTION. 1. By Corollary 1, the possible rational roots are the integral divisors of -28, or ± 1, ± 2, ± 4, ± 7, ± 14, and ± 28.

2. By synthetic division of $f(x)$ by $x - 1$, we find $f(1) = -6$; hence, 1 is not a root. Also, $f(-1) = -42$; hence, -1 is not a root.

3. By synthetic division by $x - 2$, we find $f(2) = 0$, and

$$f(x) = (x - 2)(x^3 - 4x^2 - 5x + 14). \qquad \begin{array}{|r|r|r|r|r|l} 1 & -6 & 3 & 24 & -28 & \underline{|2} \\ & 2 & -8 & -10 & 28 & \\ \hline 1 & -4 & -5 & 14 & 0 & \end{array}$$

Hence, $x = 2$ is a root. The other roots of $f(x) = 0$ are the roots of the *depressed equation* $x^3 - 4x^2 - 5x + 14 = 0$.

4. Let $Q(x) = x^3 - 4x^2 - 5x + 14$. Then, the possible rational roots of $Q(x) = 0$ are ± 1, ± 2, ± 7, ± 14. From Step 2, ± 1 are not roots. By synthetic division of $Q(x)$ by $x + 2$, we find $Q(-2) = 0$ and

$$Q(x) = (x + 2)(x^2 - 6x + 7). \qquad \begin{array}{|r|r|r|r|l} 1 & -4 & -5 & 14 & \underline{|-2} \\ & -2 & 12 & -14 & \\ \hline 1 & -6 & 7 & 0 & \end{array}$$

Hence, -2 is a root. The depressed equation is $x^2 - 6x + 7 = 0$, whose solutions, obtained by the quadratic formula, are $x = 3 \pm \sqrt{2}$, which are irrational. Hence, 2 and -2 are the only rational roots of $f(x) = 0$.

EXAMPLE 2. Find all roots of $f(x) = 3x^3 + 2x^2 - 3x - 2 = 0$.

SOLUTION. 1. By the theorem on rational roots, if c/d is a root, the possible values of c are ± 1 and ± 2; the possible values of d are ± 1 and ± 3. On forming all possible fractions c/d from these values, we find the following as the possible rational roots: ± 1; ± 2; $\pm \frac{1}{3}$; $\pm \frac{2}{3}$.

2. By synthetic division by $x - 1$, we find that $f(1) = 0$, and that

$$f(x) = (x - 1)(3x^2 + 5x + 2). \qquad \begin{array}{|r|r|r|r|l} 3 & 2 & -3 & -2 & \underline{|1} \\ & 3 & 5 & 2 & \\ \hline 3 & 5 & 2 & 0 & \end{array}$$

Hence, $x = 1$ is a root, and the depressed equation is $3x^2 + 5x + 2 = 0$. On solving this equation, we find that the other roots are -1 and $-\frac{2}{3}$.

EXERCISE 62*

Find all rational roots and, if their determination leads to a depressed equation which is a quadratic, find all the roots. In case there are no rational roots, this fact must be rigorously demonstrated.

1. $x^3 + 4x^2 + x - 6 = 0$.
2. $x^3 - 5x^2 - 8x + 12 = 0$.
3. $x^3 + 2x^2 - 9x - 18 = 0$.
4. $x^3 + x^2 - 5x + 3 = 0$.
5. $x^3 - x^2 - 3x + 6 = 0$.
6. $x^3 - 5x^2 + 5x + 3 = 0$.
7. $x^4 - 40x = 45x^2 - 84$.
8. $x^4 + x^2 + 2x + 6 = 0$.
9. $x^3 + 5x^2 + 2x - 6 = 0$.
10. $3x^3 - 10x^2 - 2x + 4 = 0$.
11. $2x^3 - x^2 - 6x - 10 = 0$.
12. $3x^3 + 2x^2 - 2x - 8 = 0$.
13. $4x^3 - 9x^2 + 14x - 3 = 0$.
14. $6x^3 - x^2 - 6x - 2 = 0$.
15. $9x^4 - 9x^3 + 5x^2 = 4 - 4x$.
16. $2x^3 - 3x^2 - x - 1 = 0$.
17. $16x^3 - 22x^2 + 5x + 1 = 0$.
18. $2x^3 + 9x^2 - 2x - 30 = 0$.

★*First find limits for the roots and then obtain all rational roots, making use of the limits and general theorems in rejecting possibilities.*

19. $x^3 + 8x^2 + 17x + 70 = 0$.
20. $x^3 - 5x^2 - 41x + 180 = 0$.
21. $x^3 + 7x^2 + 31x + 52 = 0$.
22. $x^4 + x^3 - 2x^2 = 480 - 232x$.
23. $x^5 - 6x^3 - x^2 + 60x = 54$.
24. $x^4 + 4x^3 - 30x^2 = 44x + 165$.

★143. Transformation to multiply the roots. If we substitute $x = \dfrac{y}{m}$ in $f(x) = 0$, we obtain $f\left(\dfrac{y}{m}\right) = 0$, an equation in y, each of whose roots is m times a root of $f(x) = 0$, because $y = mx$.

THEOREM I. *To obtain a simplified form of the equation* $f\left(\dfrac{y}{m}\right) = 0$, *each of whose roots is m times a root of* $f(x) = 0$, *where*

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n, \tag{1}$$

multiply the successive coefficients in $f(x) = 0$, *starting with* a_1, *by* $m, m^2, m^3, \cdots, m^n$, *respectively, and replace x by y.*

Proof. 1. On replacing x by y/m in $f(x) = 0$, we obtain

$$f\left(\frac{y}{m}\right) = a_0\left(\frac{y}{m}\right)^n + a_1\left(\frac{y}{m}\right)^{n-1} + \cdots + a_{n-1}\left(\frac{y}{m}\right) + a_n = 0. \tag{2}$$

2. On multiplying both sides of (2) by m^n, we obtain

$$a_0y^n + a_1my^{n-1} + a_2m^2y^{n-2} + \cdots + a_{n-1}m^{n-1}y + a_nm^n = 0,$$

which completes the proof of Theorem I.

* The instructor may wish to assign Section 143 before Exercise 62.

In applying Theorem I, we supply each missing power in $f(x)$ with zero as a coefficient.

ILLUSTRATION 1. To obtain an equation each of whose roots is three times a root of $2x^3 + 3x^2 - 5 = 0$, we substitute $y/3$ for x and obtain

$$2y^3 + 3 \cdot 3y^2 - 5 \cdot 3^3 = 0, \quad \text{or} \quad 2y^3 + 9y^2 - 135 = 0.$$

By use of Theorem I, we obtain a new method for finding the rational roots of an equation with integral coefficients of the form

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = 0, \quad \textit{where} \quad a_0 \neq 1.$$

EXAMPLE 1. Find the rational roots of $64x^3 - 16x^2 + 12x - 3 = 0$.

SOLUTION. 1. Divide by 64:
$$x^3 - \frac{x^2}{4} + \frac{3x}{16} - \frac{3}{64} = 0. \tag{3}$$

2. Transform (3) to multiply the roots by 4, chosen as the smallest integer which will cause the resulting new equation to be in the b-form with integral coefficients. We substitute $x = \frac{1}{4}y$ and by Theorem I we obtain

$$y^3 - y^2 + 3y - 3 = 0. \tag{4}$$

3. The only rational roots of (4) are integers; by the method of page 148 we obtain $y = 1$ as the only rational root. Since $x = \frac{1}{4}y$, the only rational root of (3) is $x = \frac{1}{4}$.

★EXERCISE 63

Transform, to multiply the roots as specified.

1. $3x^3 + 2x^2 - 5x + 3 = 0$; to multiply the roots by 2.
2. $2x^4 - 3x^2 + 5x - 7 = 0$; to multiply the roots by -2.
3. $x^4 - \frac{1}{4}x^3 + \frac{3}{8}x^2 - \frac{5}{8} = 0$; to multiply the roots by 4.
4. $x^4 + \frac{1}{3}x^2 + \frac{2}{3}x - \frac{1}{81} = 0$; to multiply the roots by 3.

Find all rational roots by the method of Example 1, Section 143.

5–12. Solve Problems 11–18 of Exercise 62 by the new method.

13. $16x^3 - 28x^2 + 9 = 0$.

14. $4x^4 - 23x^2 = 11x - 15$.

15. $8x^4 + 28x^2 = 9x - 18$.

16. $8x^4 + 36x^2 - 9x + 35 = 0$.

144. Location theorem. *If a and b are real numbers for which $f(a)$ and $f(b)$ have unlike signs, then $f(x) = 0$ has at least one root between $x = a$ and $x = b$.*

Proof. On a graph of $f(x)$, the points P and Q corresponding to $x = a$ and $x = b$ will be on opposite sides of the x-axis. Since the graph is a continuous curve joining P and Q, the graph must cross the x-axis at least once, and in any case an odd number of times,

between P and Q. To each intersection with the x-axis there corresponds a root of $f(x) = 0$.

145. Irrational roots obtained by successive graphs.*

EXAMPLE 1. Solve: $x^3 - 3x^2 - 2x + 5 = 0$.

SOLUTION. **1.** By the method of Section 142 we find that there are no rational roots. Let $f(x)$ represent the left member; we graph $f(x)$ in Figure 24 by use of the following table of values. From the graph, we read that the roots are approximately -1.3, 1.2, and 3.2.

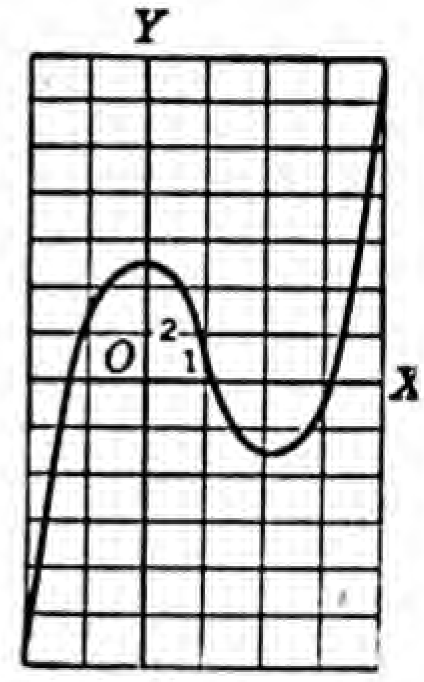


FIG. 24

WHEN $x =$	-2	-1	0	1	2	3	4
THEN $f(x) =$	-11	3	5	1	-3	-1	13

2. To obtain more accurately the root near 3.2.

A. *Locate the root between successive tenths.* We compute $f(3.2) = .65$ and $f(3.1) = -.24$. Hence, there is a root between 3.1 and 3.2.

B. *Graph $f(x)$ from $x = 3.1$ to $x = 3.2$, by use of $f(3.1)$ and $f(3.2)$.*

WHEN $x =$	3.1	3.2
THEN $f(x) =$	$-.24$	$+.65$

The graph, in Figure 25, is taken as a straight line because we use only two points to determine it. If more points were used, the graph would be curved. Figure 25 is sufficient for our purposes; in it we read that the root is approximately 3.13.

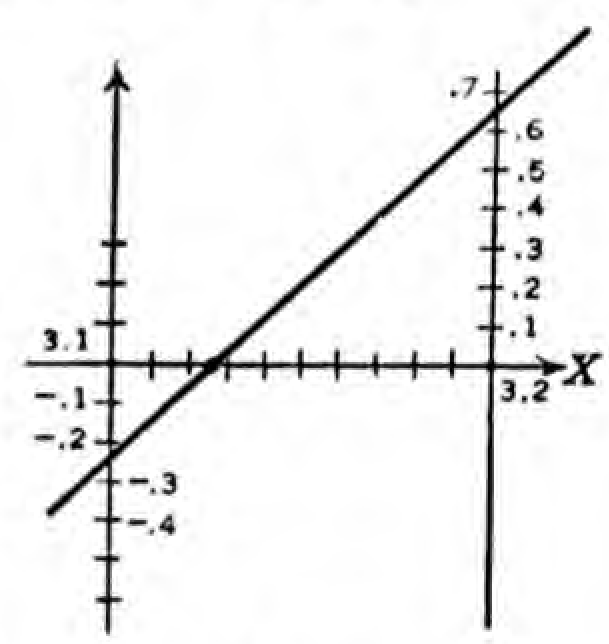


FIG. 25

C. *Locate the root between successive hundredths.* By computing the values in the table below, we find that

WHEN $x =$	3.12	3.13
THEN $f(x) =$	$-.073$	$+.014$

the root is between 3.12 and 3.13. From a graph of $f(x)$ between $x = 3.12$ and $x = 3.13$, in Figure 26, we read that the root is approximately 3.128. The final "8" is open to question, but could be verified by an additional graphical step.

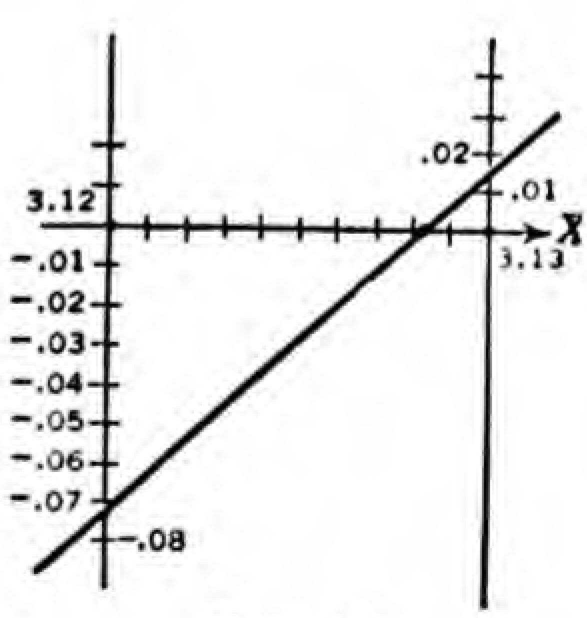


FIG. 26

3. Similarly, we find the other roots: $x = 1.202$; $x = -1.330$.

* This section may be omitted if Horner's method is to be studied.

$\frac{125}{8} + \frac{25}{4} - \frac{10}{4} + 5$

METHOD I. *To obtain an irrational root by successive enlargements of a graph.*

1. *From a graph of $f(x)$, estimate the root to the nearest tenth and call this estimated value x_1.*

2. *Locate the root between successive tenths by computing $f(x)$ for $x = x_1$ and for the successive tenths near x_1.*

3. *Graph $f(x)$, with an enlarged scale, for values of x between the tenths where the root lies, using a straight line as an approximation to the curve. From the graph, read the apparent value x_2 of the root to hundredths.*

4. *Locate the root between successive hundredths by computing values of $f(x)$. Continue, by the method of Steps 2 and 3, until the root is obtained as accurately as is desired.*

To obtain an irrational root correct to k decimal places by the preceding method, it is necessary to determine the root approximately to $k + 1$ decimal places.

★*Note 1.* The result, 3.13, obtained from Figure 25 can be found *without a figure* by use of **simple interpolation,** as used for instance with logarithm tables. To employ this method, we first notice that, if r is the unknown root, then $f(r) = 0$. From the following data, we assume that, since 0 is

WHEN $x =$	3.1	r	3.2
THEN $f(x) =$	$-.24$	0	.65

$3.2 - 3.1 = .1;\quad 0 - (-.24) = .24;$
$.65 - (-.24) = .89.$

24/89 of the *numerical distance* from $-.24$ to .65, then r is *the same proportion of the way from* 3.1 *to* 3.2. Or, we assume that

$$r = 3.1 + \frac{24}{89}(.1) = 3.1 + .027 = 3.13, \textit{ approximately.}$$

The preceding assumption is *exactly* equivalent to assuming that the graph of $f(x)$ between $x = 3.1$ and $x = 3.2$ is a *straight line:* this statement can be proved by drawing appropriate triangles in Figure 25 and then using the properties of similar triangles. The interested student should carry out this construction.

Note 2. If all roots of $a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$ are *real,* their determination can be checked by use of the fact, proved in a later section, that *the sum of the roots equals* $-\dfrac{a_1}{a_0}$. Thus, in Example 1, $-\dfrac{a_1}{a_0} = 3$ *and the sum of the roots is* $(3.128 + 1.202 - 1.330) = 3.$

146. Outline for finding the real roots* of an equation.

1. *If the coefficients are rational, find all rational roots. Let $f(x) = 0$ be the depressed equation after all rational roots are removed.*

2. *Compute $f(x)$ for convenient integral values of x and obtain a rough graph of $f(x)$.*†

3. *Read the approximate value of each real root from the graph.*

4. *By successive graphs,*‡ *obtain each irrational root in turn to the desired number of decimal places.*

EXERCISE 64§

Find the specified roots accurate to two decimal places:

1. The root of $2x^3 - 5x^2 - x + 5 = 0$ between 1 and 2.

2. The root of $x^3 - x^2 - x + 99 = 0$ between -4 and -5.

3. The root of $x^3 + 2x^2 - x - 1 = 0$ between 0 and 1.

4. The root of $x^4 - x^3 + x^2 + 2x = 6$ between -1 and -2.

5. The *two* roots of $x^3 - 3x^2 + 13 = 4x$ between 2 and 3.

Find all real roots accurate to two decimal places.

6. $x^3 - 2x^2 = 5x - 4$.

7. $x^3 - 3x^2 + 6x = 9$.

8. $x^4 - 3x^3 + x^2 + 2x = 1$.

9. $x^4 - 11x^2 - 15x = 2$.

10. $x^4 - 2x^3 - 5 = 0$.

11. $x^4 - 8.75 = 0$.

12. $4x^3 - 4x = 1$.

13. $x^3 + 2x^2 - 14x + 13 = 0$.

Find the indicated principal roots correct to two decimal places:

14. $\sqrt[4]{8}$. **15.** $\sqrt[3]{256}$. **16.** $\sqrt[3]{-37}$. **17.** $\sqrt[5]{-163}$.

HINT. The positive root of $x^4 - 8 = 0$ is the value of $\sqrt[4]{8}$.

The following equations have no rational roots; find their real roots correct to two decimal places:

18. $x^3 - 1.57x^2 + .431 = 0$.

19. $x^3 - 2.47x^2 + .733 = 0$.

20. $x^3 + 12x - 302 = 0$.

21. $x^4 + 380x - 1444 = 0$.

22. The edges of a rectangular box are 3′, 4′, and 6′ long. To double the volume, each dimension is increased by the same amount. Find the new dimensions correct to two decimal places.

* If Descartes' rule of signs has been studied, it should be applied to obtain information about the roots as the first step in the solution.

† If Section 141 has been studied, limits for the roots may be found and the graph should extend between these limits.

‡ Or, by Horner's method, if it is to be used by the class.

§ The instructor may wish to introduce Sections 147 and 148 before Exercise 64.

In the remaining problems, find all real solutions correct to two decimal places, unless otherwise stated. In problems 23 and 24, solve for x and y.

23. $\begin{cases} 2y = x^2 - 2, \\ 4y^2 - 12y - x + 8 = 0. \end{cases}$ **24.** $\begin{cases} 4y = x^2 + 4x + 4, \\ x^2 + 9y^2 = 36. \end{cases}$

25. An open box is to be made from a rectangular piece of cardboard, 15″ long and 10″ wide, by cutting equal squares from the corners and turning up the sides. Find the side of these squares if the box is to contain 122 cubic inches.

26. A spherical shell has an inner radius of 10″; the volume of the solid part is 1100 cubic inches. Find the thickness of the shell. Use $\pi = 3\frac{1}{7}$.

The depth d to which a solid floating sphere will sink in water is a positive root of the equation $d^3 - 3rd^2 + 4r^3s = 0$, where r is the radius of the sphere and s is the specific gravity of the substance composing the sphere. Find d for each sphere below:

27. Cork sphere: radius = 3′; specific gravity = .21.

28. Wooden sphere: radius = 5′; specific gravity = .71.

★29. The thermal conductivity, k, of air at an absolute temperature of $T°$ (Fahrenheit), is given by the equation

$$k = .0129 \frac{717}{T + 225}\left(\frac{T}{492}\right)^{\frac{3}{2}}.$$

By use of Section 145, find the temperature to the nearest degree if $k = .0158$. Use logarithms and do not expand or rationalize in the equation.

★30. The relation $M = x - E \sin x$, called **Kepler's equation,** holds between the *mean anomaly* M and the *eccentric anomaly* x of a planet at any point in its path around the sun; E represents the eccentricity of the path, and M and x are measured in radians. If $E = .5$ and $M = 4$, find x by the method of Section 145.

★147. Transformation to decrease the roots.* Let $f(x)$ represent the left side of $a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$. On substituting $x = X + h$ in $f(x) = 0$, we obtain $f(X + h) = 0$, or

$$a_0(X + h)^n + a_1(X + h)^{n-1} + \cdots + a_{n-1}(X + h) + a_n = 0, \quad (1)$$

whose roots are those of $f(x) = 0$ each decreased by h, because we have $X = x - h$. If the powers in (1) are expanded and like powers of X are collected, we obtain a_0X^n and meet certain coefficients $A_1, \cdots, A_n$ for the other powers:

$$f(X + h) = a_0X^n + A_1X^{n-1} + \cdots + A_{n-1}X + A_n = 0. \quad (2)$$

* Auxiliary to the discussion of Horner's method for irrational roots.

THEOREM I. *To obtain the coefficients $A_1, A_2, \cdots, A_n$ in the equation $f(X+h)=0$, whose roots are those of $f(x)=0$ each decreased by h, divide $f(x)$ by $x-h$; divide the resulting quotient by $x-h$; etc. Continue to divide each quotient by $x-h$ until n divisions have been performed. The remainder obtained in the first division is A_n; the second remainder is A_{n-1}; $\cdots$; the last remainder is A_1; the last quotient is a_0.*

Proof. 1. In (2), if we substitute $X=x-h$, then $f(X+h)$ becomes $f(x)$ and we obtain

$$f(x)=a_0(x-h)^n+A_1(x-h)^{n-1}+\cdots+A_{n-1}(x-h)+A_n. \quad (3)$$

2. If we divide $f(x)$ in (3) by $x-h$, the quotient is

$$a_0(x-h)^{n-1}+A_1(x-h)^{n-2}+\cdots+A_{n-2}(x-h)+A_{n-1}, \quad (4)$$

and the remainder is A_n. On dividing (4) by $x-h$, we obtain the new quotient

$$a_0(x-h)^{n-2}+A_1(x-h)^{n-3}+\cdots+A_{n-3}(x-h)+A_{n-2},$$

and the remainder A_{n-1}. On continuing through the n divisions specified in Theorem I, the last quotient which we obtain is a_0 and the remainders, in order, are $A_n, A_{n-1}, \cdots, A_2, A_1$.

ILLUSTRATION 1. To obtain an equation whose roots are those of $2x^3-7x+6=0$ each decreased by 2, we divide by $x-2$, synthetically, as specified in Theorem I.

	2	0	-7	6	$\underline{\vert 2}$
		4	8	2	
First quotient is $2x^2+4x+1$.	2	4	1	8	$A_3=8$.
		4	16		
Second quotient is $2x+8$.	2	8	17		$A_2=17$.
		4			
Third quotient is 2.	2	12			$A_1=12$.

The desired equation: $2X^3+12X^2+17X+8=0.$

★EXERCISE 65

Transform to decrease the roots as specified.

1. $2x^3+2x^2-5x+43=0$; to decrease the roots by 1.
2. $3x^3+4x^2-2x+5=0$; to decrease the roots by 2.
3. $x^4-3x^2+5x-7=0$; to decrease the roots by .3.
4. $2x^3+x^4-x+3=0$; to decrease the roots by -2.
5. $3x^3+2x^2-2x+5=0$; to decrease the roots by $-.03$.

★148. Horner's method for finding irrational roots. The method presented in this section applies only to integral rational equations.*

EXAMPLE 1. Solve: $$f(x) = x^3 - 3x^2 - 2x + 5 = 0. \quad (1)$$

SOLUTION. **1.** From Step 1 of the solution of Example 1 on page 151, we find that the real roots are approximately -1.3, 1.2, and 3.2.

2. To obtain the root which is between 3 and 4, proceed as follows:

A. *Transform* (1) *to decrease the roots by* 3. The transformed equation, obtained as in the preceding section, is

$$f_1(x_1) = x_1^3 + 6x_1^2 + 7x_1 - 1 = 0. \quad (2)$$

A subscript 1 on f_1 and on x_1 serves to distinguish (2) from (1). Since (1) has a root between 3 and 4, hence (2) has a root between 0 and 1. When x_1 is *small*, $(x_1^3 + 6x_1^2)$ is *very* small compared to $(7x_1 - 1)$. Hence, the root of (2) between 0 and 1 is approximately the same as the root of $7x_1 - 1 = 0$, or $x_1 = \frac{1}{7} = .1^+$. To locate this root between successive tenths, we compute values of $f_1(x_1)$ by synthetic division: we find $f_1(.1) = -.239$ and $f_1(.2) = .648$. Hence, (2) has a root between $x_1 = .1$ and $x_1 = .2$.

B. *Transform* (2) *to decrease the roots by* .1. We obtain

$$f_2(x_2) = x_2^3 + 6.3x_2^2 + 8.23x_2 - .239 = 0. \quad (3)$$

Equation 3 has a root between 0 and .1 whose approximate value is obtained from $8.23x_2 - .239 = 0$; $x_2 = .02^+$. To locate this root between successive hundredths, we compute $f_2(.02) = -.0719$ and $f_2(.03) = .0136$. Hence, the root is between .02 and .03.

C. *Transform* (3) *to decrease the roots by* .02. We obtain

$$f_3(x_3) = x_3^3 + 6.36x_3^2 + 8.4832x_3 - .0719 = 0, \quad (4)$$

which has a root between 0 and .01 whose approximate value is obtained from

$$8.4832x_3 - .0719 = 0; \; x_3 = .008^+.$$

D. *Conclusion.* Each root of (1) is 3.12 greater than a root of (4) because we successively reduced the roots by 3, .1, and .02, or, altogether, by 3.12. Since $x_3 = .008$ is approximately a root of (4), hence $x = 3.12 + .008$, or $x = 3.128$, is approximately a root of (1). This value of the root is *certainly correct to two decimal places.* If desired, the third decimal place could be checked by locating the root of (4) between successive thousandths. The essential numerical work performed in obtaining $x = 3.128$ is compactly arranged as follows:

* The method of Section 145 applies to transcendental or irrational equations as well as to integral rational equations.

1	-3	-2	5	$\lfloor 3$	($x = 3^+$.)
	3	0	-6		
1	0	-2	-1		
	3	9			
1	3	**7**			
	3				
1	**6**	**7**	$-$ **1**	$\lfloor .1$	$\left\{\begin{array}{r} 7x_1 - 1 = 0; \\ x_1 = .1^+. \end{array}\right\}$
	.1	.61	.761		
1	6.1	7.61	$-$ **.239**		
	.1	.62			
1	6.2	**8.23**			
	.1				
1	**6.3**	**8.23**	$-$ **.239**	$\lfloor .02$	$\left\{\begin{array}{r} 8.23x_2 - .239 = 0; \\ x_2 = .02^+. \end{array}\right\}$
	.02	.1264	+ .1671		
1	6.32	8.3564	$-$ **.0719**		
	.02	.1268			
1	6.34	**8.4832**			
	.02				
1	**6.36**	**8.4832**	$-$ **.0719**		($8.48x_3 - .0719 = 0;\ x_3 = .008^+$.)

Hence, $$x = 3 + .1 + .02 + .008 = 3.128.$$

3. Similarly, we find $x = 1.202$ as the second positive root.

4. To find the negative root of $f(x) = 0$, consider the equation

$$f(-x) = -x^3 - 3x^2 + 2x + 5 = 0.$$

Since $f(x) = 0$ has a root near $x = -1.3$, hence $f(-x) = 0$ has a root near $x = 1.3$. By the method of Step 2, this root of $f(-x) = 0$ is found to be $x = 1.330$. Hence, $x = -1.330$ is the negative root of $f(x) = 0$.

SUMMARY. *To find a positive irrational root by Horner's method.*

1. *Locate the root between successive integers. The smaller integer is the integral part of the root.*

2. *Transform $f(x) = 0$ into $f_1(x_1) = 0$, whose roots are those of $f(x) = 0$ decreased by the smaller of the integers of Step 1. Then, $f_1(x_1) = 0$ has a root between 0 and 1. Locate this root between successive tenths. The smaller tenth is the tenths part of the root of $f(x) = 0$.*

3. *Transform $f_1(x_1) = 0$ into $f_2(x_2) = 0$, whose roots are those of $f_1(x_1) = 0$ decreased by the smaller of the tenths found in Step 2. Then, $f_2(x_2)$ has a root between 0 and .1. Locate this root between successive hundredths. The smaller hundredth is the hundredths part of the root of $f(x) = 0$.*

4. *Continue this process, to obtain any desired accuracy. Each step yields, accurately, one more decimal place of the root.*

Note 1. To find *negative* roots of $f(x) = 0$ by Horner's method, find the *positive* roots of $f(-x) = 0$, and multiply by -1.

★149. Coefficients in terms of the roots.

ILLUSTRATION 1. Let r_1, r_2, and r_3 be the roots of

$$x^3 + b_1x^2 + b_2x + b_3 = 0. \tag{1}$$

From Section 135,

$$\begin{aligned} x^3 + b_1x^2 + b_2x + b_3 &\equiv (x - r_1)(x - r_2)(x - r_3) \\ &\equiv x^3 - (r_1 + r_2 + r_3)x^2 + (r_1r_2 + r_1r_3 + r_2r_3)x - r_1r_2r_3. \end{aligned} \tag{2}$$

From Corollary 1, page 140, it follows that

$$b_1 = -(r_1 + r_2 + r_3);\quad b_2 = r_1r_2 + r_1r_3 + r_2r_3;\quad b_3 = -r_1r_2r_3. \tag{3}$$

Similarly, we find that, if $r_1, r_2, \cdots, r_n$ are the roots of

$$x^n + b_1x^{n-1} + b_2x^{n-2} + \cdots + b_n = 0, \tag{4}$$

the following equations are true:

$$\left.\begin{aligned} b_1 &= -(r_1 + r_2 + \cdots + r_n), \\ b_2 &= r_1r_2 + r_1r_3 + \cdots + r_1r_n + r_2r_3 + \cdots + r_{n-1}r_n, \\ b_3 &= -(r_1r_2r_3 + r_1r_2r_4 + \cdots), \\ &\cdots\cdots\cdots\cdots\cdots \\ b_n &= (-1)^n r_1r_2r_3 \cdots r_n. \end{aligned}\right\} \tag{5}$$

Or, $b_1 = -$ (*the sum of all the roots*),
$b_2 = +$ (*the sum of the products of the roots, two at a time*),
$b_3 = -$ (*the sum of the products of the roots, three at a time*),
. . . .
$b_n = (-1)^n \cdot$ (*the product of the roots*).

If we divide both sides of

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0 \tag{6}$$

by a_0, we obtain

$$x^n + \frac{a_1}{a_0}x^{n-1} + \frac{a_2}{a_0}x^{n-2} + \cdots + \frac{a_n}{a_0} = 0.$$

Hence, from (5), if $(r_1, r_2, \cdots, r_n)$ are the roots of (6), then

$$\frac{a_1}{a_0} = -(r_1 + r_2 + \cdots + r_n);$$

$$\frac{a_2}{a_0} = +(r_1r_2 + r_1r_3 + \cdots);\ \text{etc.}$$

★EXERCISE 66

1. By the method of Illustration 1, page 158, derive equations 5 for the case of $x^4 + b_1x^3 + b_2x^2 + b_3x + b_4 = 0$.

By use of equations 5, *Section* 149, *obtain an equation with the given roots.*

2. $(3, -4, 2)$.
3. $(\sqrt{2}, -\sqrt{2}, -3, 1)$.
4. $(\pm 2i, 1, -5)$.

In the remaining problems, x is the unknown.

5. Find the third root of $3x^3 + 7x^2 + bx + d = 0$, given that 3 and -2 are two of the roots.

6. Find all of the roots of $3x^3 - 12x^2 + cx + f = 0$, given that one root is 4, and that the difference of the other roots is 4.

7. Solve $x^3 + 4x^2 - 9x + h = 0$, given that one root is the negative of another.

8. Solve $x^3 - 6x^2 - 4x + h = 0$, given that the roots, in some order, form an arithmetic progression.

9. Solve $x^3 - 7x^2 + 14x + k = 0$, given that the roots, in some order, form a geometric progression.

10. Solve $x^4 + 2x^3 - 11x^2 + kx + 36 = 0$, given that it has two double roots.

11. Prove that, if $x^3 + qx^2 + rx + s = 0$ has one root the negative of another, then $s = qr$, and conversely.

12. If the roots of $x^3 + bx^2 + cx + d = 0$, taken in some order, form a geometric progression, prove that $c^3 - b^3d = 0$.

★150. Algebraic solution. An algebraic formula is one involving only a finite number of the operations of addition, subtraction, multiplication, division, and the extraction of roots, and no other operations. If $f(x) = 0$ represents the general equation of degree n in a single unknown x, $a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$, then the following theorems hold.

THEOREM I. *If $n = 1$, or 2, or 3, or 4, the roots of $f(x) = 0$ can be expressed by means of algebraic formulas in terms of the coefficients of $f(x)$.*

THEOREM II. *If $n > 4$, no root of $f(x) = 0$ can be expressed by an algebraic formula in terms of the coefficients of $f(x)$.*

We have previously derived formulas for the solution of the general linear equation, $ax + b = 0$, and the general quadratic equation, $ax^2 + bx + c = 0$. In the next two sections we shall com-

plete the proof of Theorem I by deriving algebraic formulas in terms of the coefficients for the roots of the general cubic and the general quartic. The proof of Theorem II demands the methods of advanced mathematics and is beyond the scope of this text. Theorem II was first proved in 1824 by the brilliant Norwegian mathematician, Niels Henrik Abel (1802–1829), who left a tremendous record of mathematical achievement in spite of his early death, caused by tuberculosis.

★151. Solution of the general cubic* in a single unknown x,

$$x^3 + bx^2 + cx + d = 0. \tag{1}$$

Solution. 1. In (1), place
$$x = y - \frac{b}{3}. \tag{2}$$

Then, we obtain
$$y^3 + py + q = 0, \tag{3}$$

where
$$p = c - \frac{b^2}{3}, \quad \text{and} \quad q = d - \frac{bc}{3} + \frac{2b^3}{27}.$$

2. We call (3) the **reduced cubic;** its advantage is that it contains no term in y^2. In (3), place

$$y = z - \frac{p}{3z}. \tag{4}$$

Then,
$$z^3 - \frac{p^3}{27z^3} + q = 0, \quad \text{or} \quad z^6 + qz^3 - \frac{p^3}{27} = 0. \tag{5}$$

Since (5) is in the quadratic form in z^3, we solve for z^3 by use of the quadratic formula; if we let $R = \dfrac{p^3}{27} + \dfrac{q^2}{4}$, the solutions are

$$z^3 = -\frac{q}{2} + \sqrt{R} \quad \text{and} \quad z^3 = -\frac{q}{2} - \sqrt{R}. \tag{6}$$

3. The first equation in (6) has three solutions, z_1, z_2, and z_3, the cube roots of the right member. From (4), the corresponding values of y are found to be

$$y_1 = z_1 - \frac{p}{3z_1}; \quad y_2 = z_2 - \frac{p}{3z_2}; \quad y_3 = z_3 - \frac{p}{3z_3}.$$

4. Therefore, from (2), the roots of equation 1 are

$$x_1 = z_1 - \frac{p}{3z_1} - \frac{b}{3}; \quad x_2 = z_2 - \frac{p}{3z_2} - \frac{b}{3}; \quad x_3 = z_3 - \frac{p}{3z_3} - \frac{b}{3}. \tag{7}$$

Comment. It can be proved that, if in place of (z_1, z_2, z_3) we had used the solutions (z_4, z_5, z_6) of the second equation in (6), the three values obtained for x would be the same as those in (7).

* Section 127 is a prerequisite for the remainder of the chapter.

Note 1. In the solution, p, q, and R are rational functions of b, c, and d. Hence, we should think of z_1, z_2, and z_3 as the indicated cube roots of a certain function of b, c, and d. Therefore, in (7) we have algebraic formulas for x_1, x_2, and x_3 in terms of the coefficients (b, c, d).

ILLUSTRATION 1. To solve $x^3 + 3x^2 - 9x - 3 = 0$, we obtain $p = -12$, $q = 8$, and $R = -48$. Then, the first equation in (6) becomes

$$z^3 = -4 + 4i\sqrt{3} = 8(\cos 120^\circ + i \sin 120^\circ).$$

Hence, we obtain

$$z_1 = 2(\cos 40^\circ + i \sin 40^\circ); \qquad z_2 = 2(\cos 160^\circ + i \sin 160^\circ);$$
$$z_3 = 2(\cos 280^\circ + i \sin 280^\circ).$$

From (7),

$$x_1 = 2(\cos 40^\circ + i \sin 40^\circ) + \frac{12}{6(\cos 40^\circ + i \sin 40^\circ)} - 1;$$

$$x_1 = 2(\cos 40^\circ + i \sin 40^\circ) + 2(\cos 40^\circ - i \sin 40^\circ) - 1 = 4 \cos 40^\circ - 1$$

Similarly, $x_2 = 4 \cos 160^\circ - 1$; $x_3 = 4 \cos 280^\circ - 1$. From Table IV,

$$x_1 = 2.06; \quad x_2 = -4.76; \quad x_3 = -.30.$$

Note 2. The essential elements of the preceding solution of the general cubic were first published in 1545 by H. CARDAN (1501–1576) in his famous treatise called *Ars Magna*, although he had obtained the method under promise of secrecy from NICCOLÒ TARTAGLIA (1506–1557). The clue to the method is supposed to have been discovered independently by Tartaglia and an earlier writer, perhaps SCIPIONE DEL FERRO (1496–1526). Expressions 7 are frequently called Cardan's formulas, which is an injustice to the memory of Tartaglia. All mathematicians just mentioned were Italians. The French algebraist, FRANCIS VIETA (1540–1603), was the first to consider the general cubic, in contrast to special cubics.

★152. Ferrari's solution of the general quartic

$$x^4 + bx^3 + cx^2 + dx + e = 0. \tag{1}$$

SOLUTION. 1. From (1), $$x^4 + bx^3 = -cx^2 - dx - e. \tag{2}$$

We shall determine what expression should be added to both sides of (2) to make each side a perfect square. The left member contains the first two terms of $(x^2 + \frac{1}{2}bx)^2$. Hence, on adding $\frac{1}{4}b^2x^2$ to both sides of (2), we obtain

$$(x^2 + \tfrac{1}{2}bx)^2 = (\tfrac{1}{4}b^2 - c)x^2 - dx - e. \tag{3}$$

2. Let y be a number to be specified later. If $y(x^2 + \frac{1}{2}bx) + \frac{1}{4}y^2$ is added to both sides of (3), we obtain

$$(x^2 + \tfrac{1}{2}bx + \tfrac{1}{2}y)^2 = (\tfrac{1}{4}b^2 - c + y)x^2 + (\tfrac{1}{2}by - d)x + (\tfrac{1}{4}y^2 - e). \tag{4}$$

The right side of (4) is a quadratic in x which will be a perfect square if its discriminant is zero, that is, if

$$(\tfrac{1}{2}by - d)^2 - 4(\tfrac{1}{4}b^2 - c + y)(\tfrac{1}{4}y^2 - e) = 0. \tag{5}$$

3. On simplifying (5) we obtain

$$y^3 - cy^2 + (bd - 4e)y - b^2e + 4ce - d^2 = 0. \tag{6}$$

Equation 6 is called the **resolvent cubic** for equation 1; (6) can be solved by the method of Section 151.

4. Let $y = S$ be any root of (6), and substitute $y = S$ in (4). The right side becomes the square of some linear function, $hx + k$, and (4) gives

$$(x^2 + \tfrac{1}{2}bx + \tfrac{1}{2}S)^2 = (hx + k)^2. \tag{7}$$

From (7),
$$x^2 + \tfrac{1}{2}bx + \tfrac{1}{2}S = hx + k, \tag{8}$$
or
$$x^2 + \tfrac{1}{2}bx + \tfrac{1}{2}S = -(hx + k). \tag{9}$$

Each of (8) and (9) can be solved by use of the quadratic formula. The four values of x obtained on solving (8) and (9) are the roots of (1).

Comment. It can be proved that if any root of (6) other than $y = S$ were used in (7), the same four values of x would be obtained.

Note 1. By use of the formulas of Section 151, we could write a formula for the root S, of the resolvent cubic, which we used in (7). This formula would involve the coefficients (b, c, d, e) of (1). Using this formula for S, we could obtain formulas for h and k, in terms of (b, c, d, e); and, solving (8) and (9) in terms of h, k, and S, we could obtain formulas for the four roots of (1), in terms of (b, c, d, e). These formulas would be very complicated, and are of interest mainly for theoretical reasons.

Usually, the real roots of particular cubic or quartic equations are found more conveniently by Horner's method, or some other method of successive approximations rather than by use of the formulas of Sections 151 and 152.

Note 2. The preceding method was invented by the Italian mathematician, LODOVICO FERRARI (1522–1560), a pupil of Cardan, who published the method for the first time in his *Ars Magna*.

★EXERCISE 67

Solve each equation by taking the steps used in arriving at the formulas in the text, but do not employ these formulas themselves. In Problems 5–8, try to use a rational root of the resolvent cubic.

1. $x^3 + 9x + 26 = 0.$

2. $x^3 - 3x^2 - 12x + 140 = 0.$

3. $x^3 + 6x^2 - 36x - 24 = 0.$

4. $x^3 - 6x^2 + 9x - 1 = 0.$

5. $x^4 - 2x^3 + x^2 + 3 = 0.$

6. $x^4 - 4x^3 + 4x^2 + 4x = 5.$

7. $x^4 + 9x^3 + 23x^2 + 3x = 36.$

8. $x^4 - 4x^3 - 4x^2 = 42 - 37x.$

CHAPTER THIRTEEN

Computation and Logarithms

153. Significant digits. In this chapter we shall assume that any number referred to is written in decimal notation. In any number, we may visualize an endless sequence of zeros at the right of the last digit which is not zero, if there is such a last digit; such a number is called a **terminating decimal.**

ILLUSTRATION 1. $35.673 = \dfrac{35673}{1000}$. $.0001 = \dfrac{1}{10000} = \dfrac{1}{10^4} = 10^{-4}$. $\frac{3}{4} = .75000 \cdots$ is a terminating decimal. $\frac{1}{3} = .333 \cdots$ is an endless repeating decimal. $\pi = 3.14159 \cdots$ is an endless but not a repeating decimal.

In any number N, let us read its digits from left to right. Then, by definition, the **significant digits** or **figures** of N are its digits, in sequence, starting with the first one not zero and ending with the last one definitely specified. Notice that this definition does not involve any reference to the position of the decimal point in N. Usually we do not mention *final zeros* at the right in referring to the significant digits of N, except when it is the approximate value of some item of data.

ILLUSTRATION 2. The significant digits of 410.58 or of .0041058 are (4, 1, 0, 5, 8).

154. Approximate values. If T is the *true* value and A is an *approximate* value of a quantity, we agree to call $A - T$ the *error* of A.

ILLUSTRATION 1. If $T = 35.62$, and if $A = 35.60$ is an approximation to T, then the error of A is $35.60 - 35.62$, or $-.02$.

The significant digits in an approximate value A should indicate the maximum possible error of A. This error is understood to be *at most one half of a unit in the last significant place in A*, or, which is the same, *not more than 5 units in the next place to the right.*

ILLUSTRATION 2. If a surveyor measures a distance as 256.8 yards, he should mean that the error is at most .05 yard and that the true result lies between 256.75 and 256.85, since the error might be $\pm .05$.

In referring to the significant digits of an *approximate* value A, *it is essential to mention all final zeros designated in* A.

ILLUSTRATION 3. To state that a measured weight is 35.60 pounds should mean that the true weight differs from 35.60 pounds by at most .005 pound. To state that the weight is 35.6 pounds should mean that the true weight differs from this by at most .05 pound. Thus, there is a great distinction between 35.6 and 35.60 as *approximate values* although there is no difference between 35.6 and 35.60 as *abstract numbers.*

For abbreviation, or to indicate how many digits in a large number are significant, it is sometimes convenient to write a number N as *the product of an integral power of* 10 *and a number equal to or greater than* 1 *but less than* 10, *with as many significant digits as are justified by the data.*

ILLUSTRATION 4. $385{,}720 = 3.8572(100{,}000) = 3.8572(10^5)$.
$.000'000'368 = 3.68(.000'000'1) = 3.68(10^{-7})$.

ILLUSTRATION 5. If 5,630,000 is an approximate value, its appearance fails to show how many zeros are significant. If five digits are significant, we write $5.6300(10^6)$, and, if just three are significant, $5.63(10^6)$.

155. Rounding off numbers. In referring to a *place* in a number, we shall mean any place where a significant digit stands. In referring to a *decimal place,* the word *decimal* will be explicitly used.

To round off N to k figures, or to write a k-place approximation for N, means to write an approximate value with k significant digits so that the error of this value is not more than one half of a unit in the kth place, or 5 units in the first neglected place.

ILLUSTRATION 1. The seven-place approximation to π is 3.141593. On rounding off to five places, we obtain $\pi = 3.1416$; we changed 5 to 6 in the 4th decimal place because .000093 is greater than .00005.

ILLUSTRATION 2. In rounding off 315.475 to five figures, with equal justification we could specify either 315.47 or 315.48 as the result. As an arbitrary rule in this book, whenever we meet the necessity for such a choice between an odd and an even digit, where both are equally justifiable, **we agree to select the even digit.** Thus, we choose 315.48 here.

156. Accuracy of computation. By illustrations, we can verify that the following rules do not *underestimate* the accuracy of computation with approximate values. On the other hand, we must admit that the rules sometimes *overestimate* the accuracy. How-

ever, we shall assume that a result obtained by these rules will have a negligible error in the last significant place which is specified.

I. *In adding approximate values, round off the result in the first place where the last significant digit of any given value is found.*

II. *In multiplying or dividing approximate values, round off the result to the smallest number of significant figures found in any given value.*

ILLUSTRATION 1. Let $a = 35.64$, $b = 342.72$, and $c = .03147$ be approximate values. Then, $a + b + c$ is not reliable beyond the *second* decimal place because both a and b are subject to an unknown error which may be as large as 5 units in the third decimal place. Hence, we write

$$a + b + c = 378.39147 = 378.39, \textit{ approximately.}$$

ILLUSTRATION 2. If $x = 31.27$ and $y = .021$ are approximate values, then, by Rule II, we state that $xy = .66$, because y has only two significant digits:

$$xy = 31.27(.021) = .65667 = .66, \textit{ approximately.}$$

ILLUSTRATION 3. If a surveyor measures a rectangular field as 385.6′ by 432.4′, it would give an unjustified appearance of accuracy to write that the area is (385.6)(432.4) = 166,733.44 square feet. For, an error of .05 foot in either dimension would cause an error of about 20 square feet in the area. A reasonably justified result would be that the area is 166,700 square feet, to the nearest 100 square feet, or $1.667(10^5)$ square feet, where we claim just four significant digits; this corresponds with Rule II.

In problems where approximate values enter, or where approximate results are obtained from exact data, the results should be rounded off so as to avoid giving a false appearance of accuracy. No hard and fast rules for such rounding off should be adopted, and the final decision as to the accuracy of a result should be made only after a careful examination of the details of the solution.

EXERCISE 68

Express as a power of 10.

1. 1,000,000. **2.** 10,000. **3.** .001. **4.** .1. **5.** 1. **6.** .00001.

Round off, first to five and then to three significant digits.

7. 15.32573. **9.** .3142678. **11.** 195.635. **13.** .0345645.
8. .00132146. **10.** 5.62153. **12.** .00129553. **14.** 392,462.

Write the number in ordinary decimal form.

15. $3.85(10^4)$. **16.** $2.672(10^3)$. **17.** $1.935(10^{-4})$. **18.** $8.107(10^{-6})$.

Write as the product of a power of 10 *and a number between* 1 *and* 10.

19. 3,807,000. **20.** 67,030,000. **21.** .000357. **22.** .00000046.

If the measured length of a line is given as the specified number of feet, tell between what two values the exact length lies.

23. 567. **24.** 567.4. **25.** 567.0. **26.** 35.18. **27.** 8.139. **28.** 21.30.

Assuming that the numbers represent approximate data, find their sum and their product, and state the results without false accuracy.

29. 31.52 and .0186. **30.** .023424 and 1.13. **31.** 3.6 and .21573.

32. The measured dimensions of a rectangular field are 469.2 feet and 57.3 feet. Find the perimeter and the area of the field.

Given that the number is an approximate value, write it as the product of an integral power of 10 *and a number between* 1 *and* 10, *assuming, first, that there are five and, second, that there are three significant digits.*

33. 9,325,000. **34.** 460,000. **35.** 23,500,000. **36.** 72,200,000.

157. Irrational exponents. A logical foundation for the use of irrational exponents is beyond the scope of this text. Hence, without discussion, we shall assume the fact that irrational powers have meaning and that the laws of exponents hold if the exponents involved are real numbers, either *rational* or *irrational,* provided that the base is *positive.*

ILLUSTRATION 1. The student may safely use his intuition in connection with the symbol $10^{\sqrt{2}} = 10^{1.414\cdots}$. Closer and closer approximations to $10^{\sqrt{2}}$ are obtained if the successive decimal approximations to $\sqrt{2}$ are used as exponents. That is, $10^{\sqrt{2}}$ can be approximated as closely as we please if we proceed far enough out in the sequence

$$10^1,\ 10^{1.4},\ 10^{1.41},\ 10^{1.414},\ \cdots.$$

158. Logarithms are auxiliary numbers which are *exponents,* and which permit us to simplify the arithmetical operations of multiplication, division, raising to powers, and extraction of roots. In the following definition, a represents any *positive number, not* 1, and N is any *positive number.*

DEFINITION I. *The logarithm of a number N to the base a is the exponent of the power to which a must be raised to obtain N.*

In other words, if $a^x = N$ then x is the logarithm of N to the base a. To abbreviate "*the logarithm of N to the base a,*" we write "$\log_a N$." Then, by Definition I, the following equations state

the same fact, the first equation in exponential form and the second in logarithmic form:

$$N = a^x \quad and \quad x = \log_a N. \tag{1}$$

Illustration 1. If $N = 4^5$, then 5 is the logarithm of N to the base 4.

Illustration 2. "$\log_2 64$" is read "*the logarithm of* 64 *to the base* 2":

$$since \quad 64 = 2^6, \quad hence \quad \log_2 64 = 6.$$

Illustration 3. Since $\sqrt[3]{5} = 5^{\frac{1}{3}}$, hence $\log_5 \sqrt[3]{5} = \frac{1}{3} = .333 \cdots$.

Illustration 4. To find $\log_2 \frac{1}{8}$, we express $\frac{1}{8}$ as a power of 2:

$$since \quad \frac{1}{8} = \frac{1}{2^3} = 2^{-3}, \quad hence \quad \log_2 \frac{1}{8} = -3.$$

Illustration 5. If $\log_b 16 = 4$, then $b^4 = 16$; $b = \sqrt[4]{16} = 2$.

Illustration 6. If $\log_a 2 = -\frac{1}{3}$, then $a^{-\frac{1}{3}} = 2$. Hence,

$$\frac{1}{a^{\frac{1}{3}}} = 2; \quad a^{\frac{1}{3}} = \frac{1}{2}; \quad a = \left(\frac{1}{2}\right)^3 = \frac{1}{8}.$$

For any base a, we have $a^0 = 1$ and $a^1 = a$. Hence,

$$\log_a 1 = 0; \qquad \log_a a = 1. \tag{2}$$

In advanced mathematics, it is proved that, if $N > 0$ and $a > 0$, there exists just one real number x such that $N = a^x$. That is, *every positive number N has just one real logarithm to the base a.*

Note 1. We do not use $a = 1$ as a base for logarithms because every power of 1 is 1 and hence no number except 1 could have a logarithm to the base 1.

Note 2. We shall not define or use logarithms of *negative* numbers. If $N < 0$, or if $a < 0$, $\log_a N$ can be defined as a complex number.

EXERCISE 69

Express as a fraction.

1. a^{-5}. **2.** b^{-3}. **3.** 2^{-4}. **4.** 3^{-4}. **5.** 10^{-2}. **6.** 10^{-3}.

Express by use of a fractional exponent.

7. $\sqrt{a}$. **8.** $\sqrt[3]{b}$. **9.** $\sqrt[3]{5}$. **10.** $\sqrt[4]{8}$. **11.** $\sqrt[5]{10}$. **12.** $\sqrt[6]{10}$.

By use of equations 1, *write an equivalent logarithmic equation.*

13. $N = 3^5$. **16.** $N = 5^{\frac{1}{4}}$. **19.** $64 = 4^3$. **22.** $\frac{1}{25} = 5^{-2}$.

14. $N = 10^4$. **17.** $N = 10^{-.4}$. **20.** $64 = 8^2$. **23.** $\frac{1}{32} = 2^{-5}$.

15. $N = 10^{-3}$. **18.** $49 = 7^2$. **21.** $27 = 3^3$. **24.** $\frac{1}{100} = 10^{-2}$.

Find the number N whose logarithm is given.

25. $\log_5 N = 2$.
26. $\log_8 N = 2$.
27. $\log_6 N = 3$.
28. $\log_4 N = \frac{1}{2}$.
29. $\log_{10} N = 0$.
30. $\log_9 N = 1$.
31. $\log_4 N = -2$.
32. $\log_{10} N = -3$.
33. $\log_8 N = \frac{1}{3}$.
34. $\log_9 N = \frac{1}{2}$.
35. $\log_{125} N = \frac{1}{3}$.
36. $\log_{81} N = \frac{3}{2}$.

Find the following logarithms.

37. $\log_6 36$.
38. $\log_2 32$.
39. $\log_4 2$.
40. $\log_{10} 100$.
41. $\log_3 27$.
42. $\log_6 216$.
43. $\log_{100} 10{,}000$.
44. $\log_9 3$.
45. $\log_{100} 10$.
46. $\log_8 \frac{1}{8}$.
47. $\log_3 \frac{1}{81}$.
48. $\log_{10} .0001$.

★*Find a, N, or x, whichever is not given.*

49. $\log_a 16 = 2$.
50. $\log_a 125 = 3$.
51. $\log_a 1000 = 3$.
52. $\log_a 9 = \frac{1}{2}$.
53. $\log_a 3 = \frac{1}{3}$.
54. $\log_a 10 = \frac{1}{4}$.
55. $\log_a \frac{1}{25} = -1$.
56. $\log_a .001 = -3$.
57. $\log_{81} N = \frac{3}{2}$.
58. $\log_{100} N = \frac{5}{2}$.
59. $\log_{.01} N = -\frac{1}{2}$.
60. $\log_{10} .1 = x$.
61. $\log_a .0001 = -2$.
62. $\log_{625} 25 = x$.
63. $\log_a 8 = -\frac{3}{2}$.

159. Properties of logarithms useful in computation.

I. *The logarithm of a product equals the sum of the logarithms of its factors. For instance,*

$$\log_a MN = \log_a M + \log_a N. \tag{1}$$

ILLUSTRATION 1. $\text{Log}_{10} (897)(596) = \log_{10} 897 + \log_{10} 596$.

Proof. Let $x = \log_a M$, and $y = \log_a N$. Then,

$$M = a^x, \quad \text{and} \quad N = a^y. \qquad \text{(Definition of a logarithm)}$$

$$MN = a^x a^y = a^{x+y}. \qquad \text{(A law of exponents)}$$

Therefore, by Definition I, $\log_a MN = x + y = \log_a M + \log_a N$.

Note 1. By use of (1) we can prove Property I for a product of any number of factors. Thus, since $MNP = (MN)(P)$,

$$\log_a MNP = \log_a MN + \log_a P = \log_a M + \log_a N + \log_a P.$$

II. *The logarithm of a quotient equals the logarithm of the dividend minus the logarithm of the divisor:*

$$\log_a \frac{M}{N} = \log_a M - \log_a N. \tag{2}$$

ILLUSTRATION 2. $\text{Log}_{10} \frac{89}{57} = \log_{10} 89 - \log_{10} 57$.

Proof of Property II. Let $x = \log_a M$, and $y = \log_a N$; then,

$$M = a^x \quad \text{and} \quad N = a^y.$$

Therefore, $$\frac{M}{N} = \frac{a^x}{a^y} = a^{x-y}. \qquad \text{(A law of exponents)}$$

Hence, $$\log_a \frac{M}{N} = x - y = \log_a M - \log_a N. \qquad \text{(Definition I)}$$

III. *The logarithm of the kth power of a number N equals k times the logarithm of N:*

$$\log_a N^k = k \log_a N. \tag{3}$$

Proof. Let $x = \log_a N$. Then, by Definition I, $N = a^x$.

Therefore, $$N^k = (a^x)^k = a^{kx}. \qquad \text{(A law of exponents)}$$

Hence, by Definition I, $$\log_a N^k = kx = k \log_a N.$$

ILLUSTRATION 3. $\log_a 7^5 = 5 \log_a 7$. $\quad \log_a \sqrt[4]{3} = \log_a 3^{\frac{1}{4}} = \frac{1}{4} \log_a 3$.

Since $\sqrt[h]{N} = N^{\frac{1}{h}}$, by use of (3) with $k = 1/h$ we obtain

$$\log_a \sqrt[h]{N} = \frac{1}{h} \log_a N. \tag{4}$$

ILLUSTRATION 4. $\log_a \sqrt{N} = \frac{1}{2} \log_a N$; $\quad \log_a \sqrt[3]{25} = \frac{1}{3} \log_a 25$.

160. Logarithms to the base 10 are called **common logarithms** and are the most useful variety for computational purposes. Hereafter, unless otherwise stated, when we mention a *logarithm* we shall mean a *common* logarithm. For abbreviation, we shall write merely $\log N$, instead of $\log_{10} N$, for the common logarithm of N. The following common logarithms will be useful later; the student should obtain them by use of Definition I.

$N =$	.0001	.001	.01	.1	1	10	100	1000	10,000	100,000
$\log N =$	-4	-3	-2	-1	0	1	2	3	4	5

ILLUSTRATION 1. If we are given $\log 3 = .4771$, then by use of Properties I, II, and III we obtain the following results:

$$\log 300 = \log 3(100) = \log 3 + \log 100 = .4771 + 2 = 2.4771;$$

$$\log .003 = \log \frac{3}{1000} = \log 3 - \log 1000 = .4771 - 3 = -2.5229;$$

$$\log \sqrt[4]{3} = \log 3^{\frac{1}{4}} = \frac{1}{4} \log 3 = \frac{1}{4}(.4771) = .1193.$$

EXERCISE 70

Find the common logarithm of each number by use of Properties I, II, *and* III *and the following logarithms.*

$$\log 2 = .3010;\ \log 3 = .4771;\ \log 7 = .8451;\ \log 17 = 1.2304.$$

1. 6.	**5.** $\frac{7}{3}$.	**9.** .7.	**13.** 9.	**17.** $\sqrt[3]{7}$.	**21.** $\sqrt[3]{14}$.
2. 34.	**6.** $\frac{17}{7}$.	**10.** .17.	**14.** 49.	**18.** $\sqrt[4]{51}$.	**22.** $\sqrt[4]{21}$.
3. 30.	**7.** $\frac{7}{2}$.	**11.** $\frac{51}{2}$.	**15.** 27.	**19.** $\frac{1}{3}$.	**23.** $\sqrt{\frac{2}{7}}$.
4. 70.	**8.** $\frac{10}{7}$.	**12.** $\frac{3}{14}$.	**16.** 8.	**20.** $\frac{1}{7}$.	**24.** $\sqrt[3]{\frac{3}{17}}$.

Prove by the method used in establishing Properties I, II, *and* III, *without using the properties themselves.*

25. $\log_a MNP^2 = \log_a M + \log_a N + 2 \log_a P.$

26. $\log_a \dfrac{M}{NP} = \log_a M - \log_a N - \log_a P.$

27. $\log_a M^3\sqrt{N} = 3 \log M + \frac{1}{2} \log N.$

161. Characteristic and mantissa. The logarithm of any number can be written as the sum of an integer and a decimal fraction, positive or zero and less than 1. After $\log N$ is written in this way, we call the integer the **characteristic** and the fraction the **mantissa** of $\log N$. We notice that the characteristic of $\log N$ is *negative* when and only when $\log N$ itself is a *negative* number.

$$\textbf{log } N = \textbf{(an integer)} + \textbf{(a fraction, } \geqq 0, < 1\textbf{)};$$
$$\textbf{log } N = \textbf{characteristic} + \textbf{mantissa}. \qquad (1)$$

ILLUSTRATION 1. If $\log N = 4.6832 = 4 + .6832$, then .6832 is the mantissa and 4 is the characteristic of $\log N$.

ILLUSTRATION 2. If $\log N = -3.75$, then $\log N$ lies between -4 and -3. Hence, $\log N = -4 + (\text{a fraction})$. To find the fraction we subtract: $4 - 3.75 = .25$. Hence, $\log N = -3.75 = -4 + .25$; the characteristic is -4 and the mantissa is .25.

ILLUSTRATION 3. The following logarithms were obtained by later methods. The student should verify the three columns at the right.

	LOGARITHM	CHARACTERISTIC	MANTISSA
$\log 300 = 2.4771$	$= 2 + .4771$	2	.4771
$\log 50 = 1.6990$	$= 1 + .6990$	1	.6990
$\log .001 = -3$	$= -3 + .0000$	-3	.0000
$\log 6.5 = 0.8129$	$= 0 + .8129$	0	.8129
$\log .0385 = -1.4145$	$= -2 + .5855$	-2	.5855

162. Properties of the characteristic and mantissa.

ILLUSTRATION 1. All numbers whose logarithms are given below have the same significant digits, (3, 8, 0, 4). To obtain the logarithms, log 3.804 was first found from a table to be discussed later; the other logarithms were then obtained by use of Properties I and II.

$$\begin{aligned}
\log 380.4 &= \log 100(3.804) = \log 100 + \log 3.804 = 2 + .5802;\\
\log 38.04 &= \log 10(3.804) = \log 10 + \log 3.804 = 1 + .5802;\\
\log 3.804 &= .5802 = 0 + .5802;\\
\log .3804 &= \log \frac{3.804}{10} = \log 3.804 - \log 10 = -1 + .5802;\\
\log .03804 &= \log \frac{3.804}{100} = \log 3.804 - \log 100 = -2 + .5802.
\end{aligned}$$

Similarly, if N is *any* number whose significant digits are (3, 8, 0, 4), then N equals 3.804 multiplied, or else divided, by a positive integral power of 10; hence, it follows as before that .5802 is the mantissa of $\log N$.

Note 1. Let $x = \log M$ and $y = \log N$; then $M = 10^x$ and $N = 10^y$. For integral values of x and y we have observed that if $x < y$ then $10^x < 10^y$, and conversely. This relation extends to the case where x and y are not necessarily integral; that is,

$$\log M < \log N \quad \textit{if and only if} \quad M < N. \tag{2}$$

In Illustration 1, the characteristic of log 380.4 is 2, of log 38.04 is 1, etc. These facts could have been learned as follows.

ILLUSTRATION 2. To find the characteristic of log 380.4, notice the two successive integral powers of 10 between which 380.4 lies:

$$100 < 380.4 < 1000.$$

Hence,

$$\log 100 < \log 380.4 < \log 1000; \quad \text{or,} \quad 2 < \log 380.4 < 3.$$

Therefore, $\log 380.4 = 2 + (\text{a fraction}, > 0, < 1)$; or, by definition, the characteristic of log 380.4 is 2.

In Illustration 1 we met special cases of the following theorems.

THEOREM I. *The mantissa of* $\log N$ *depends only on the sequence of significant digits in* N. *That is,* **if two numbers differ only in the position of the decimal point, their logarithms have the same mantissa.**

THEOREM II. **When $N > 1$,** *the characteristic of* $\log N$ *is an integer, positive or zero, which is* **one less than the number of digits in N to the left of the decimal point.**

THEOREM III. **If** $N < 1$, *the characteristic of* log N *is a negative integer;* **if the first significant digit of N is in the kth decimal place, then $-k$ is the characteristic of log N.**

ILLUSTRATION 3. By use of Theorems II and III, we find the characteristic of log N by merely inspecting N. Thus, by Theorem III, the characteristic of log .00039 is -4 because "3" is in the 4th decimal place. By Theorem II, the characteristic of log 15,786 is 4, because there are five digits to the left of the decimal point (which is understood after 6).

Note 2. Besides common logarithms, the only other variety used appreciably is the system of **natural,** or **Naperian logarithms,** for which the base is a certain irrational number denoted by e where $\boldsymbol{e = 2.71828 \cdots}$. Natural logarithms are useful for theoretical purposes.

Note 3. Logarithms were invented by a Scotchman, JOHN NAPIER, Baron of Merchiston (1550–1617). His original logarithms were not the same as those now called Naperian logarithms, in his honor. Common logarithms, also called Briggs logarithms, were invented by an Englishman, HENRY BRIGGS (1556–1631), who was aided by Napier.

163. Standard form for a negative logarithm. Hereafter, for convenience in computation, **if the characteristic of log N is negative, $-k$, change it to the equivalent value**

$$[(10 - k) - 10], \quad \text{or} \quad [(20 - k) - 20], \text{ etc.}$$

ILLUSTRATION 1. Given that log .000843 $= -4 + .9258$, we write

$$\log .000843 = -4 + .9258 = (6 - 10) + .9258 = 6.9258 - 10.$$

The characteristics of the following logarithms are obtained by use of Theorem III; the mantissas are identical, by Theorem I.

1ST SIGNIF. DIGIT IN	ILLUSTRATION	LOG N	STANDARD FORM
1st *decimal place*	$N = .843$	$-1 + .9258$	$= 9.9258 - 10$
2d *decimal place*	$N = .0843$	$-2 + .9258$	$= 8.9258 - 10$
6th *decimal place*	$N = .00000843$	$-6 + .9258$	$= 4.9258 - 10$

164. A table of logarithms. Mantissas can be computed by use of advanced mathematics, and, except in special cases, are endless decimal fractions. Computed mantissas are found in *tables of logarithms,* also called *tables of mantissas.*

ILLUSTRATION 1. The mantissa for log 10705 is .029586671630457, to fifteen decimal places.

Table II gives the mantissa of log N correct to four decimal places if N has at most three significant digits; a decimal point is understood in front of each mantissa in the table. If N lies between 1 and 10, the characteristic of log N is zero so that log N is *the same as its mantissa.* Hence, a four-place table of *mantissas* is also a table of the *actual logarithms of all numbers with at most three significant digits,* from $N = 1.00$ to $N = 10.00$. In case N is less than 1 or greater than 10, we must supply the characteristic of log N by use of Theorems II and III besides obtaining the mantissa of log N by use of Table II.

EXAMPLE 1. Find log .0316 from Table II.

SOLUTION. 1. *To obtain the mantissa:* find "31" in the column headed N in the table; in the row for "31," read the entry in the column headed "6." The mantissa is .4997.

2. By Theorem III, the characteristic of log .0316 is -2, or $(8 - 10)$. Hence,
$$\log .0316 = -2 + .4997 = 8.4997 - 10.$$

ILLUSTRATION 2. From Table II and Theorem II, $\log 31{,}600 = 4.4997$.

EXAMPLE 2. Find N if $\log N = 7.6064 - 10$.

SOLUTION. 1. *To find the significant digits of N:* the mantissa of log N is .6064; this is found in Table II as the mantissa for the digits "404."

2. *To locate the decimal point in N:* the characteristic of log N is $(7 - 10)$ or -3; hence, by Theorem III, $N = .00404$.

ILLUSTRATION 3. If $\log N = 3.6064$, the characteristic is 3 and, by Theorem II, N has 4 figures to the left of the decimal point: the mantissa is the same as in Example 2. Hence, $N = 4040$.

DEFINITION I. *A number N is called the* **antilogarithm** *of L in case* $\log N = L$, *and for abbreviation we write* $N =$ **antilog** L.

ILLUSTRATION 4. Since $\log 1000 = 3$, hence $1000 = \text{antilog } 3$.

ILLUSTRATION 5. Example 2 could have been stated as follows: *find antilog* $(7.6064 - 10)$.

★165. A five-place table of logarithms* lists the mantissa of log N correct to five decimal places if N is any number with at most four significant digits.

* The reader is referred to the five-place table in HART'S *Logarithmic and Trigonometric Tables* or in HART'S *Tables for Mathematics of Investment, 3d Edition,* D. C. HEATH AND COMPANY, publishers. The exercises are arranged to permit the use of either four-place or five-place tables.

EXAMPLE 1. Find log .03162 from a five-place table.

SOLUTION. 1. *To obtain the mantissa.* We find the first three digits, 316, under the column headed "N." In the row of 316, in the column headed by 2, the fourth digit of 3162, we find 996, the last three digits of the mantissa; its first two digits are 49, shown in the column headed by 0. The mantissa is .49996.

2. By Theorem III, the characteristic is -2. Hence,

$$\log .03162 = -2 + .49996 = 8.49996 - 10.$$

EXAMPLE 2. Find N if $\log N = 5.40209 - 10$.

SOLUTION. 1. *To obtain the significant digits of N.* The mantissa is .40209. We find its first two digits "40" in the column headed by 0. Among the entries corresponding to this "40" we find 209 in the row with 252 at the left margin and in the column headed by 4. Hence, the significant digits of N are 2524.

2. The characteristic of $\log N$ is $(5 - 10)$ or -5; by Theorem III, the first significant digit of N is in the 5th decimal place: $N = .00002524$.

EXERCISE 71

In Problems 1 to 8, each number is the logarithm of some number N. State the characteristic and the mantissa of $\log N$.

1. 2.9356. **3.** 15.2162. **5.** -1.300. **7.** $7.2356 - 10$.

2. 3.5473. **4.** $-2 + .3561$. **6.** -5.675. **8.** $5.1942 - 10$.

Write the following negative logarithms in standard form.

9. $-1 + .2562$. **10.** $.3267 - 3$. **11.** $.4932 - 6$. **12.** -3.4675.

State the characteristic of the logarithm of each number.

13. 637,500. **14.** 368. **15.** .000673. **16.** .00897. **17.** .000007.

Use Table II *to find the four-place logarithm of the number.*

18. 65.4. **21.** .00785. **24.** 6530. **27.** .086. **30.** .00089.

19. 43.2. **22.** .0346. **25.** 17,800. **28.** .000358. **31.** 157,000.

20. 178. **23.** 9.46. **26.** .00005. **29.** 101,000. **32.** .0000002.

Find the antilogarithm of the given logarithms by use of Table II.

33. 2.3856. **37.** 0.1553. **41.** $9.4800 - 10$. **45.** $8.9823 - 10$.

34. 3.3927. **38.** 2.1461. **42.** $8.5611 - 10$. **46.** $4.8915 - 10$.

35. 3.6684. **39.** 1.8692. **43.** $7.7701 - 10$. **47.** $6.9542 - 10$.

36. 1.8785. **40.** 0.9727. **44** $9.8041 - 10$. **48.** $2.9340 - 10$.

49. Find N if (*a*) $\log N = -2.3010$; (*b*) $\log N = 8.3010 - 10$.

50. Find N if (*a*) $\log N = -3.6990$; (*b*) $\log N = 7.6990 - 10$.

★*Find the five-place logarithm of each number.*

51. 198.7.	**55.** .01118.	**59.** 59,600.	**63.** .801.	**67.** 1,000,000.
52. 18.56.	**56.** .2866.	**60.** 69,990.	**64.** 3.075.	**68.** .000607.
53. 1.389.	**57.** .2563.	**61.** .00018.	**65.** 4168.	**69.** $10^{.69897}$.
54. 2.633.	**58.** .0146.	**62.** .00009.	**66.** 10,070.	**70.** $10^{2.41326}$.

★*Find the antilogarithms of the following five-place logarithms.*

71. 1.25115.	**75.** 9.42716 − 10.	**79.** 0.66058.	**83.** 6.55630 − 10.
72. 2.47305.	**76.** 8.58726 − 10.	**80.** 5.83052.	**84.** 5.68124 − 10.
73. 4.68538.	**77.** 7.49094 − 10.	**81.** 3.61899.	**85.** 0.11361.
74. 3.77663.	**78.** 9.09237 − 10.	**82.** 0.48001.	**86.** 2.30081.

166. Interpolation in a table of mantissas is based on the assumption that, *for small changes in N, the corresponding changes in* $\log N$ *are proportional to the changes in N.* This **principle of proportional parts** is merely an approximation to the truth but leads to results which are sufficiently accurate for our purposes.

We agree that, whenever a mantissa is found by interpolation from a table, we shall express the result *only to the number of decimal places given in table entries.* Also, in finding N by interpolation in a table of mantissas when $\log N$ is given, we agree to specify just **four** or just **five** significant digits according as we are using a **four-place** or a **five-place** table. No greater refinement in the result is justified because the unavoidable error, which may arise, frequently will be as large as 1 unit in the last significant digit which we have agreed to specify, although the error is rarely larger.

167. Interpolation in a four-place table.

EXAMPLE 1. Find log 13.86 by interpolation in Table II.

SOLUTION. 1. We notice that $13.80 < 13.86 < 13.90$. Hence, by the principle of proportional parts, we assume that, since 13.86 is $\frac{6}{10}$ of the way from 13.80 to 13.90,

$$\log 13.86 \textit{ is } \tfrac{6}{10} \textit{ of the way from } \log 13.80 \textit{ to } \log 13.90, \textit{ or}$$

$$\log 13.86 = \log 13.80 + .6(\log 13.90 - \log 13.80).$$

2. Each logarithm below has 1 for its characteristic, by Theorem II.

From table:	log 13.80 = 1.1399	31	**Tabular difference** *is*
	log 13.86 = ?		.1430 − .1399 = .0031.
From table:	log 13.90 = 1.1430		.6(.0031) = .00186, or .0019.

$$\log 13.86 = 1.1399 + .6(.0031) = 1.1399 + .0019 = 1.1418.$$

Comment. We found $.6(31) = 18.6$ by use of the table headed "31" under the *column of proportional parts* in Table II.

ILLUSTRATION 1. To find log .002914:

$$10\left[\begin{array}{l} 4\left[\begin{array}{l} 2910:\ \textit{mantissa is } .4639 \\ 2914:\ \textit{mantissa is } \ ? \end{array}\right] x \\ \quad 2920:\ \textit{mantissa is } .4654 \end{array}\right] 15$$

Tabular difference *is*
$.4654 - .4639 = .0015.$
$x = .4(15) = 6.$

Hence, the mantissa for 2914 *is* $.4639 + .0006 = .4645$.

Hence, by Theorem III, $\log .002914 = -3 + .4645 = 7.4645 - 10$.

EXAMPLE 2. Find N if $\log N = 1.6187$.

SOLUTION. 1. The mantissa .6187 is not in Table II but lies between the consecutive entries .6180 and .6191, the mantissas for 415 and 416.

2. Since .6187 is $\frac{7}{11}$ of the way from .6180 to .6191, we assume that N is $\frac{7}{11}$ of the way from 41.50 to 41.60.

$$11\left[\begin{array}{l} 7\left[\begin{array}{l} 1.6180 = \log 41.50 \\ 1.6187 = \log N \end{array}\right] x \\ \quad 1.6191 = \log 41.60 \end{array}\right] .10$$

$41.60 - 41.50 = .10$
$x = \frac{7}{11}(.10) = .064$, *or approximately* .06.

$$N = 41.50 + \tfrac{7}{11}(.10) = 41.50 + .06 = 41.56.$$

ILLUSTRATION 2. To find N if $\log N = 6.1053 - 10$:

$$34\left[\begin{array}{l} 15\left[\begin{array}{l} .1038,\ \textit{mantissa for } 1270 \\ .1053,\ \textit{mantissa for } \ ? \end{array}\right] x \\ \quad .1072,\ \textit{mantissa for } 1280 \end{array}\right] 10$$

$\frac{15}{34} = .4$. *Hence,*
$x = .4(10) = 4.$
$1270 + 4 = 1274.$

Hence, .1053 *is the mantissa for* 1274 *and* $N = .0001274$.

Comment. We obtain $\frac{15}{34} = .4$ by inspection of the tenths of 34 in the columns of proportional parts. We read

$$13.6 = .4(34) \quad \text{or} \quad \frac{13.6}{34} = .4, \quad \text{and} \quad \frac{17}{34} = .5.$$

Since 15 is nearer to 13.6 than to 17, hence $\frac{15}{34}$ is nearer to .4 than to .5.

If $N = P(10^k)$ where k is an integer and P is greater than or equal to 1 but less than 10, then $\log P$ *is the mantissa and* k *is the characteristic of* $\log N$.

ILLUSTRATION 3. If $N = 1.352(10^8)$ then, by Property I, page 168,

$$\log N = \log 1.352 + \log 10^8 = 0.1309 + 8 = 8.1309.$$

If $N = 1.352(10^{-4})$, then $\log N = .1309 - 4 = 6.1309 - 10$.

ILLUSTRATION 4. If $\log N = 9.7419$, and if we write N in the form $P(10^k)$, where $1 \leqq P < 10$, we observe that $k = 9$ and $\log P = 0.7419$:

$$P = 5.520 \quad \text{and} \quad N = 5.520(10^9).$$

This form *shows explicitly that only four digits are significant in the result.*

★168. Interpolation in a five-place table.

EXAMPLE 1. Find log 25.637 from a five-place table.

SOLUTION. By the principle of proportional parts, log 25.637 is $\frac{7}{10}$ of the way from log 25.630 to log 25.640.

$$\begin{array}{ll} \textit{From table:} & \left.\begin{array}{l} \log 25.630 = 1.40875 \\ \log 25.637 = \quad ? \\ \log 25.640 = 1.40892 \end{array}\right] 17 \\ \textit{From table:} & \end{array} \qquad \begin{array}{c} \textbf{Tabular difference } \textit{is} \\ .40892 - .40875 = .00017. \\ .7(17) = 11.9, \text{ or } 12. \end{array}$$

$$\log 25.637 = 1.40875 + .7(.00017) = 1.40875 + .00012 = 1.40887.$$

EXAMPLE 2. Find N if $\log N = 2.40971$.

SOLUTION. The mantissa is .40971, which, numerically, is between the consecutive entries .40960 and .40976 in the five-place table; these mantissas correspond to 2568 and 2569. Since .40971 is $\frac{11}{16}$ of the way from .40960 to .40976, we assume that the significant part of N is $\frac{11}{16}$ of the way from 25680 to 25690.

$$16\left[\begin{array}{l} 11\left[\begin{array}{l} .40960, \textit{ mantissa for } 25680 \\ .40971, \textit{ mantissa for } \quad ? \end{array}\right] x \\ \quad .40976, \textit{ mantissa for } 25690 \end{array}\right] 10 \qquad \begin{array}{c} \frac{11}{16} = .7, \textit{ to nearest tenth.} \\ x = .7(10) = 7. \\ 25680 + 7 = 25687. \end{array}$$

Hence, .40971 *is the mantissa for* 25687 and $N = 256.87$.

Note 1. In interpolating in a table of mantissas, if there is equal reason for choosing either of two successive digits, for uniformity we agree to make that choice which gives an **even digit** in the last significant place of the *final result* of the interpolation.

EXERCISE 72

Find the four-place logarithm of each number from Table II.

1. 1826.	**5.** 35.94.	**9.** .5627.	**13.** 90,090.	**17.** $1.233(10^{-4})$.
2. 25.63.	**6.** 1.293.	**10.** .03147.	**14.** 203,500.	**18.** $1.417(10^{6})$.
3. 532.2.	**7.** .3013.	**11.** .01563.	**15.** .001439.	**19.** $3.126(10^{6})$.
4. 12.67.	**8.** .4213.	**12.** .001139.	**16.** .05626.	**20.** $2.438(10^{-2})$.

Find the antilogarithm of each four-place logarithm from Table II.

21. 3.2367. **22.** 7.1247 − 10. **23.** 6.1640. **24.** 8.9935 − 10.

25. 3.1395.
26. 2.9276.
27. 1.6016.
28. 0.4906.
29. 6.3350 − 10.
30. 4.1436 − 10.
31. 9.6715 − 10.
32. 8.0255 − 10.
33. 3.8862.
34. 2.1952.
35. 0.0130
36. 5.5511.
37. 5.9885 − 10.
38. 8.3358 − 20.
39. 9.6270 − 10.
40. 0.4228.

★*Find the five-place logarithm of each number.*

41. 18,563.
42. 25,632.
43. 5.3217.
44. 21.285.
45. .30129.
46. .042087.
47. 4.7178.
48. 31.648.
49. .073563.
50. .89316.
51. .75362.
52. 53.193.
53. .0040063.
54. .0062873.
55. .00078651.
56. 1,300,600.
57. 966,910.
58. .00041569.
59. .000000000061.
60. 5,000,600,000.

★*Find the antilogarithm of each five-place logarithm.*

61. 2.21388.
62. 3.21631.
63. 1.33740.
64. 2.05297.
65. 9.65328 − 10.
66. 8.12277 − 10.
67. 7.94014 − 10.
68. 9.77817 − 10.
69. 6.03271.
70. 5.45698.
71. 0.97036.
72. 0.28779.
73. 9.00858 − 10.
74. 3.33412 − 10.
75. 6.24049 − 20.
76. 8.73168 − 20.

169. Computation of products and quotients. Unless otherwise specified, we shall assume that the data of any given problem are *exact*. Under this assumption, the accuracy of a product, quotient, or power computed by use of logarithms depends on the number of places in the table being used. The result is frequently subject to an unavoidable error which usually is at most a few units in the last significant place given by interpolation. Hence, usually we must compute with at least *five-place* logarithms to obtain *four-place accuracy*, and with at least *four-place* logarithms to obtain *three-place accuracy*. As a general custom, in any result we shall give *all digits obtainable by interpolation* in the specified table.

EXAMPLE 1. Compute .0631(7.208)(.5127) by use of Table II.

SOLUTION. Let P represent the product. By Property I, we obtain $\log P$ by adding the logarithms of the factors. We obtain the logarithms of the factors from Table II, add to obtain $\log P$, and then finally obtain P from Table II. The computing form, given in blackface type, was made up completely as *the first step in the solution.*

$$
\begin{array}{rll}
\log \mathbf{.0631} = & 8.8000 - 10 & \text{(Table II)} \\
\log \mathbf{7.208} = & 0.8578 & \text{(Table II)} \\
\log \mathbf{.5127} = & 9.7099 - 10 & \text{(Table II)} \\
\hline
(\textit{add}) \log \mathbf{P} = & 19.3677 - 20 = 9.3677 - 10. &
\end{array}
$$

Hence, P = .2332. [= antilog (9.3677 − 10), Table II]

EXAMPLE 2. Compute $q = \dfrac{4.803 \times 269.9 \times 1.636}{7880 \times 253.6}$.

SOLUTION. First make a computing form, as given in blackface type. By Property II, $\log q =$ (*log of numerator*) $-$ (*log of denominator*).

$$(+)\begin{cases} \log \mathbf{4.803} = 0.6815 \\ \log \mathbf{269.9} = 2.4312 \\ \log \mathbf{1.636} = 0.2138 \end{cases} \qquad\qquad (+)\begin{cases} \log \mathbf{7880} = 3.8965 \\ \log \mathbf{253.6} = 2.4041 \end{cases}$$

$$\textbf{log denom.} = 6.3006.$$

$$\begin{array}{rcl} \textbf{log numer.} &= 3.3265 =& 13.3265 - 10 \\ (-)\ \textbf{log denom.} &= 6.3006 =& 6.3006 \\ \hline \log q = &\ ?\ =& 7.0259 - 10. \end{array}$$

Hence, $q = .001061$.

Comment. We foresaw that $\log q$ would be negative. To obtain $\log q$ in the standard form, we *added* 10 to 3.3265 and then *subtracted* 10. When it is necessary to subtract one logarithm from a smaller one, *increase the characteristic of the smaller one by* 10 *and then subtract* 10, *to compensate.*

EXAMPLE 3. Compute the reciprocal of 189 by use of Table II.

SOLUTION. Let $R = \dfrac{1}{189}$.

$$\begin{array}{rcl} \log 1 &= 0.0000 =& 10.0000 - 10 \\ (-)\ \log \mathbf{189} &= 2.2765 =& 2.2765 \\ \hline \longleftarrow \log R = &\ ?\ =& 7.7235 - 10. \end{array}$$

Hence, $R = .005290$.

Comment. In stating $R = .005290$, **the final zero is essential.**

Note 1. Before finding the *four*-place $\log N$ if N has more than four significant digits, round off N to *four* significant digits. Similarly, round off N to *five* significant digits if the *five*-place $\log N$ is to be used.

EXERCISE 73

Compute by use of four-place or five-place logarithms.

1. $31.57 \times .789$. **3.** $.8475 \times .0937$. **5.** $925.618 \times .000217$.

2. $925.6 \times .137$. **4.** $.0179 \times .35641$. **6.** $3.41379 \times .0142$.

7. $(-84.75)(.00368)(.02458)$. **8.** $(-16.8)(136.943)(.00038)$.

HINT. Only *positive* numbers have *real* logarithms. First compute as if all factors were positive; then determine the sign by inspection.

9. $\dfrac{675}{13.21}$. **11.** $\dfrac{728.72}{895}$. **13.** $\dfrac{.0894}{.6358}$. **15.** $\dfrac{1}{325.932}$.

10. $\dfrac{568.5}{23.14}$. **12.** $\dfrac{753.166}{9273.8}$. **14.** $\dfrac{.0421}{.53908}$. **16.** $\dfrac{1}{100{,}935}$.

17. $\dfrac{16.083 \times 256}{47 \times .0158}$. **19.** $\dfrac{9.32 \times 531}{.8319 \times .5685}$. **21.** $\dfrac{1}{.53819 \times .0673}$.

18. $\dfrac{.42173 \times .217}{.3852 \times .956}$. **20.** $\dfrac{5.4171 \times .429}{18.1167 \times 37}$. **22.** $\dfrac{1}{.00073 \times .965}$.

23. $\dfrac{(-.29)(.038)(-.0065)}{(-1006.332)(2.71)}$. **24.** $\dfrac{(5.6)(-3.9078)(-.00031)}{(132)(-1.93)}$.

Compute the reciprocal of the number.

25. 63283. **26.** .00382. **27.** .02567. **28.** .0683(.52831).

29. (*a*) Compute 652(735); (*b*) compute (log 652)(log 735).

30. (*a*) Compute (.063)(.952); (*b*) compute (log .063)(log .952).

31. (*a*) Compute .351 ÷ 625; (*b*) compute (log .351) ÷ (log 625).

★170. Cologarithms.* The logarithm of the *reciprocal* of N is called the *cologarithm* of N and is written **colog** N. Since $\log 1 = 0$,

$$\textbf{colog}\ N = \log \frac{1}{N} = 0 - \log N. \tag{1}$$

Illustration 1. Colog .031 = $\log \dfrac{1}{.031}$:

$$\begin{array}{rl} \log 1 = & 10.0000 - 10 \\ (-)\ \log .031 = & 8.4914 - 10 \\ \hline \text{colog } .031 = & 1.5086. \end{array}$$

The positive part of colog N can be quickly obtained by inspection of log N: *subtract each digit (except the last) in the positive part of log N from* 9, *and subtract the last digit from* 10.

Example 1. Compute $q = \dfrac{16.083 \times 256}{47 \times .0158}$ by use of cologarithms.

Solution. To *divide* by N is the same as to *multiply* by $1/N$. Hence, instead of *subtracting the logarithm* of each factor of the denominator, we *add the cologarithm* of the factor:

$$q = \frac{16.083 \times 256}{47 \times .0158} = (16.083 \times 256)\left(\frac{1}{47}\right)\left(\frac{1}{.0158}\right).$$

$$\begin{array}{lrl} & \log 16.08 = & 1.2063 \\ & \log 256 = & 2.4082 \\ \log 47 \quad = 1.6721;\ \text{hence,} & \text{colog } 47 = & 8.3279 - 10 \\ \log .0158 = 8.1987 - 10;\ \text{hence,} & \text{colog } .0158 = & 1.8013 \\ \hline q = 5542. \longleftarrow & (add)\ \log q = & 13.7437 - 10 = 3.7437. \end{array}$$

171. Computation of powers and roots.

Example 1. Compute $(.3156)^4$.

Solution. $\log (.3156)^4 = 4 \log (.3156) = 4(9.4991 - 10).$
$\log (.3156)^4 = 37.9964 - 40 = 7.9964 - 10.$

Therefore, $(.3156)^4 = .009918.$

* The instructor may desire to specify the use of cologarithms occasionally.

EXAMPLE 2. Compute $\sqrt[6]{.08351}$.

SOLUTION. By equation 4, page 169, $\log \sqrt[6]{N} = \frac{1}{6} \log N$.

$$\log \sqrt[6]{.08351} = \frac{\log .08351}{6} = \frac{8.9218 - 10}{6};$$

$$\log \sqrt[6]{.08351} = \frac{58.9218 - 60}{6} = 9.8203 - 10. \qquad (1)$$

$$\text{Therefore, } \sqrt[6]{.08351} = .6611.$$

Comment. Before dividing a negative logarithm by a positive integer, usually it is best to write the logarithm in such a way that the negative part, after division, will be -10. Thus, in (1), we altered $(8.9218 - 10)$ by *subtracting* 50 from -10 to make it -60, and by *adding* 50 to 8.9218 to compensate for the subtraction; the result after division by 6 is in the standard form for a negative logarithm.

EXAMPLE 3. Compute $q = \left(\dfrac{(.5831)^3}{65.3\sqrt{146}}\right)^{\frac{2}{5}}$.

SOLUTION. 1. Let F represent the fraction. Then $\log q = \frac{2}{5} \log F$.

2. Notice that $\log (.5831)^3 = 3 \log .5831$; $\log \sqrt{146} = \frac{1}{2} \log 146$.

$$\begin{array}{rl} \log .5831 = & 9.7658 - 10 \\ \log 146 = & 2.1644 \\ \hline 3 \log .5831 = & 9.2974 - 10 \\ (-)\ \log \text{denom.} = & 2.8971 \\ \hline \log F = & 6.4003 - 10; \end{array} \qquad (+)\left\{\begin{array}{rl} \log 65.3 = & 1.8149 \\ \frac{1}{2} \log 146 = & 1.0822 \end{array}\right. \atop \overline{\log \text{denom.} = 2.8971.}$$

$$2 \log F = 2.8006 - 10 = 42.8006 - 50.$$

$$\log q = \frac{2 \log F}{5} = \frac{42.8006 - 50}{5} = 8.5601 - 10. \qquad \textbf{Hence, } q = .03632.$$

EXERCISE 74

Compute by use of four-place or five-place logarithms, as the instructor directs.

1. $(17.5)^3$.	**7.** $\sqrt[3]{.857}$.	**13.** $(700{,}928)^{\frac{1}{4}}$.	**19.** $(357)^{\frac{2}{3}}$.
2. $(3.1279)^4$.	**8.** $\sqrt[4]{.03107}$.	**14.** $\sqrt[6]{1.045}$.	**20.** $(-.00831)^3$.
3. $(.837)^5$.	**9.** $(1.04)^7$.	**15.** $\sqrt[5]{.0001}$.	**21.** $(143.54)^{\frac{3}{4}}$.
4. $(.0315)^3$.	**10.** $(10{,}000)^{\frac{1}{3}}$.	**16.** $\sqrt[3]{.00001}$.	**22.** $\sqrt{\sqrt[5]{.847}}$.
5. $\sqrt{1.09}$.	**11.** $(.0797)^{\frac{1}{5}}$.	**17.** $(-1.03)^{\frac{1}{3}}$.	**23.** $(-.0057)^{\frac{2}{5}}$.
6. $\sqrt[3]{2795}$.	**12.** $(.0138273)^{\frac{1}{2}}$.	**18.** $(-1796)^{\frac{1}{5}}$.	**24.** $(157)^{-3}$.

HINT for Problem 24. Recall that $(157)^{-3} = 1/(157^3)$.

25. $(13.67)^{\frac{2}{7}}$.	**27.** $(.98)^{-2}$.	**29.** $(1.03)^{-5}$.	**31.** $(1.04)^{100}$.
26. $(3.035)^{-4}$.	**28.** $(.831447)^{-5}$.	**30.** $(1.05)^{-6}$.	**32.** $(1.04)^{-50}$.

Note 1. Give the seven-place log 1.04 = 0.0170333, compute the results in Problems 31 and 32 and compare with the former answers. The new results are more accurate (see the Answer Book).

Compute by use of four-place or five-place logarithms.

33. $.958(12.167)^2$. **34.** $10^{1.65}\sqrt{8.265}$. **35.** $10^{2.36}\sqrt{.68147}$. **36.** $(25.3)^2\sqrt[3]{.093}$.

37. $\dfrac{56.3 \times 4.317}{21.4\sqrt{521.923}}$. **41.** $\sqrt{\dfrac{89.1}{163 \times .62}}$. **45.** $\dfrac{\sqrt[3]{-463.19}}{\sqrt{16.3144}}$.

38. $\dfrac{(25.73)(152)^3}{1893.32}$. **42.** $\sqrt[3]{\dfrac{47.5317}{.031 \times .964}}$. **46.** $\sqrt[3]{\dfrac{(-316)(.198)}{.756392}}$.

39. $\dfrac{.0198}{(3.82616)^2}$. **43.** $\dfrac{10^{-.36}\sqrt{.78}}{(.983174)^2}$. **47.** $\left(\dfrac{54.2\sqrt{1.89}}{.157386}\right)^{\frac{1}{2}}$.

40. $\dfrac{758.32}{(46.3)^3}$. **44.** $\dfrac{10^{-1.42}\sqrt{.387}}{(57)(8.64)^2}$. **48.** $\left(\dfrac{5731.84}{14.2\sqrt{.896}}\right)^{\frac{3}{4}}$.

Note 2. Observe that no property of logarithms is available to simplify the computation of a *sum*. In the following problems, logarithms should be used wherever possible.

49. $\dfrac{(35.6)^2 + 89.532}{\sqrt{57} + 2.513}$. **51.** $\dfrac{\sqrt{23} - \sqrt{134.91}}{453 \times .110173}$. **53.** $\dfrac{\log 86}{\log 53.8}$.

50. $\dfrac{\sqrt[3]{45} - 364.1}{(.9873)^2 + 16.3}$. **52.** $\dfrac{(1.03)^5 + 1}{(1.03)^{\frac{1}{2}} + 1}$. **54.** $\dfrac{\log 567 - 20}{\log 235}$.

55. $(2.67)^{1.62}$. **56.** $(53.17)^{.84}$. **57.** $(59.2)^{-.43}$. **58.** $(.065)^{.532}$.

HINT for Problem 57. $-.43 \log 59.2 = -.7621 = 9.2379 - 10.$

59. Compute (*a*) $(\text{antilog } 2.6731)^2$; (*b*) $[\text{antilog } (-1.4973)]^3$.

DEFINITION I. *The* **geometric mean** *of n numbers is defined as the* nth *root of the product of the numbers.* Thus, the geometric mean of $M, N, P, Q,$ and R is $\sqrt[5]{MNPQR}$.

In each problem, find the geometric mean of the given numbers.

60. 138; 395; 426; 537; 612. **61.** .00138; .19276; .08356; .0131.

If a, b, and c are the three sides of a triangle, it is proved in trigonometry that A, the area of the triangle, is given by

$$A = \sqrt{S(S-a)(S-b)(S-c)}, \text{ where } S = \tfrac{1}{2}(a+b+c).$$

Find the area of a triangle whose sides are as follows.

62. 375.40; 141.37; 451.20. **63.** .089312; .0739168; .024853.

64. Given that $u_1, u_2, u_3, \cdots$ are positive and that $(u_1, u_2, u_3, \cdots)$ form a G.P., prove that $(\log u_1, \log u_2, \log u_3, \cdots)$ form an A.P.

The time t in seconds for one oscillation of a simple pendulum whose length is l centimeters, is given by the formula

$$t = \pi\sqrt{\frac{l}{g}}. \qquad (g = 980;\ \pi = 3.1416)$$

65. (*a*) Find the time for one oscillation of a simple pendulum .985 centimeters long. (*b*) Find l if the time for one oscillation is 3.75 seconds.

66. Let d be the diameter in inches of a short solid circular steel shaft which is designed to transmit safely H horsepower when revolving at R revolutions per minute. A safe value for d is

$$d = \sqrt[3]{\frac{38H}{R}}.$$

Find the number of horsepower which can be safely transmitted at 1150 revolutions per minute if $d = 1.9834$.

67. The weight w, in pounds of steam per second, which will flow through a hole whose cross-section area is A square inches, if the steam approaches the hole under a pressure of P pounds per square inch, is approximately $w = .0165AP^{.97}$. (It is assumed that the steam flows into a reservoir where the pressure is $\leqq \frac{3}{5}P$.) How much steam at a pressure of 83.85 pounds per square inch will flow through a hole 12.369 inches in diameter?

68. If a volume v_1 of air at a pressure p_1 is compressed to a volume v_2 at a pressure p_2, without losing any of the heat thus generated, then

$$p_2 = p_1\left(\frac{v_1}{v_2}\right)^{1.41}.$$

Under this condition, find the pressure necessary to compress 117 cubic feet of air at a pressure of 25.8 pounds per square inch to 26 cubic feet.

★172. Exponential and logarithmic equations. A *logarithmic equation* is one in which there appears the logarithm of some expression involving the unknown quantity.

EXAMPLE 1. Solve for x: $\qquad \log x + \log \frac{2x}{5} = 6.$

SOLUTION. By use of Properties I and II of page 168,

$$\log x + \log 2 + \log x - \log 5 = 6.$$

$$2 \log x = 6 + \log 5 - \log 2 = 6.3980. \qquad \text{(Table II)}$$

$$\log x = 3.1990;\ x = \text{antilog } 3.1990 = 1581. \qquad \text{(Table II)}$$

An equation where the unknown quantity appears in an exponent is called an *exponential equation*. Sometimes, an exponential equation can be solved by taking the logarithm of each member and equating results.

EXAMPLE 2. Solve $16^x = 74$.

SOLUTION. Equate the logarithms of the two sides: $x \log 16 = \log 74$;

$$x = \frac{\log 74}{\log 16} = \frac{1.8692}{1.2041}. \qquad \begin{array}{rl} & \log 1.869 = 0.2716 \\ (-) & \underline{\log 1.204 = 0.0806} \\ & \log x = 0.1910; \end{array} \qquad \text{hence} \quad x = 1.552.$$

★EXERCISE 75

Solve for x by use of four-place or five-place logarithms.

1. $12^x = 28$.
2. $51^x = 569$.
3. $5^{2x} = 28(2^x)$.
4. $15^{3x} = 85(3^x)$.
5. $.67^x = 8$.
6. $.093^x = 12$.
7. $27^{x^2} = 54$.
8. $(1.03)^{-x} = .587$.
9. $(1.04)^{-x} = .642$.
10. $6.52^x = 38.683$.
11. $6^{x^2-2x} = 18$.
12. $5^{x^2+4x} = 17.64$.
13. $\log x^2 - \log \dfrac{3x}{7} = 8.43$.
14. $\log 5x^2 + \log \dfrac{4}{x} = 5.673$.
15. $\dfrac{(1.05)^x - 1}{.05} = 6.3282$.
16. $\dfrac{(1.03)^x - 1}{.03} = 12.675$.

17. The vapor pressure p, in millimeters of mercury, of liquid arsenic trioxide is given * by $T \log p = -2722 + 6.513T$, where $T°$ Centigrade is the absolute temperature (absolute zero is $-273°$ Centigrade). (*a*) Find the pressure when $T = 500°$. (*b*) Find T if p is 120 millimeters?

18. A given radioactive substance decomposes at such a rate that, if B is the initial number of atoms of the substance and N is the number remaining at the end of t hours, then $N = Be^{-kt}$, where k is a constant and $e = 2.71828 \cdots$. Given that, out of 21,000 atoms, 14,500 remain at the end of $\frac{1}{2}$ hour, (*a*) find k; (*b*) find when only $\frac{1}{2}$ of the atoms will remain.

19. The intensity, I, of a beam of light after passing through t centimeters of a liquid which absorbs light, is given by $I = Ae^{-kt}$, where A is the intensity of the light when it enters the liquid, $e = 2.71828 \cdots$, and k is a constant for any given liquid. If $k = .12526$, find how many centimeters of the liquid are sufficient (*a*) to reduce the intensity of a beam to $\frac{1}{4}$ of its original value; (*b*) to absorb 10% of the intensity.

★173. Logarithms where the base is not 10. If N and b are given and if $x = \log_b N$, we can find x by solving $N = b^x$ for x by use of *common* logarithms.

Note 1. Recall that the base of the natural system of logarithms is $e = 2.71828$, approximately. The following logarithms are useful:

$$\log_{10} e = 0.43429; \qquad \log_{10} .43429 = 9.63778 - 10.$$

* See FARRINGTON DANIELS' *Mathematical Preparation for Physical Chemistry*, page 68. McGraw-Hill Book Company.

ILLUSTRATION 1. To find $\log_e 35$, let $x = \log_e 35$; then $e^x = 35$. On solving by the method of Section 172, we find

$$x \log_{10} e = \log_{10} 35; \qquad x = \frac{\log_{10} 35}{\log_{10} e} = \frac{1.5441}{.4343} = 3.555. \qquad \text{(Table II)}$$

THEOREM I. *If a and b are any two bases, then*

$$\log_a N = (\log_a b)(\log_b N). \tag{1}$$

Proof. Let $y = \log_b N$; then $N = b^y$. Hence,

$$\log_a N = \log_a b^y = y \log_a b = (\log_a b)(\log_b N).$$

The number $\log_a b$ is called the **modulus** of the system of base a with respect to the system of base b. Given a table of logarithms to the base b, we could form a table of logarithms to the base a by multiplying each entry of the given table by the modulus, $\log_a b$.

★EXERCISE 76

Find each logarithm by use of four-place or five-place common logarithms.

1. $\log_e 75$. **3.** $\log_{15} 33$. **5.** $\log_e 10$. **7.** $\log_5 .097$. **9.** $\log_{.8} 23.8$.

2. $\log_e 1360$. **4.** $\log_{12} 1000$. **6.** $\log_6 1.05$. **8.** $\log_e .001$. **10.** $\log_{.5} 185$.

11. Find the natural logarithm of (*a*) 4368.1; (*b*) 4.3681.

Comment on Problem 11. It is found that if two numbers differ only in the position of the decimal point, their natural logarithms do *not* differ by an *integer*. This fact, as compared to Theorem I, page 171, causes us to prefer common logarithms for aid in computation.

12. Find the modulus of each system with respect to the other: the Briggs system and the natural system of logarithms.

13. Prove that $\log_a b = 1/\log_b a$. State this result in words.

★174. Graphs of logarithmic and exponential functions. We recall that $y = \log_a x$ and $x = a^y$ are equivalent relations. We call $\log_a x$ a *logarithmic function* of x and a^y an *exponential function* of y.

Illustration 1. In Figure 27, we have the graph of $y = \log_e x$. For any base $a > 1$, the graph of $\log_a x$ would be similar. This graph assists us in remembering the following facts.

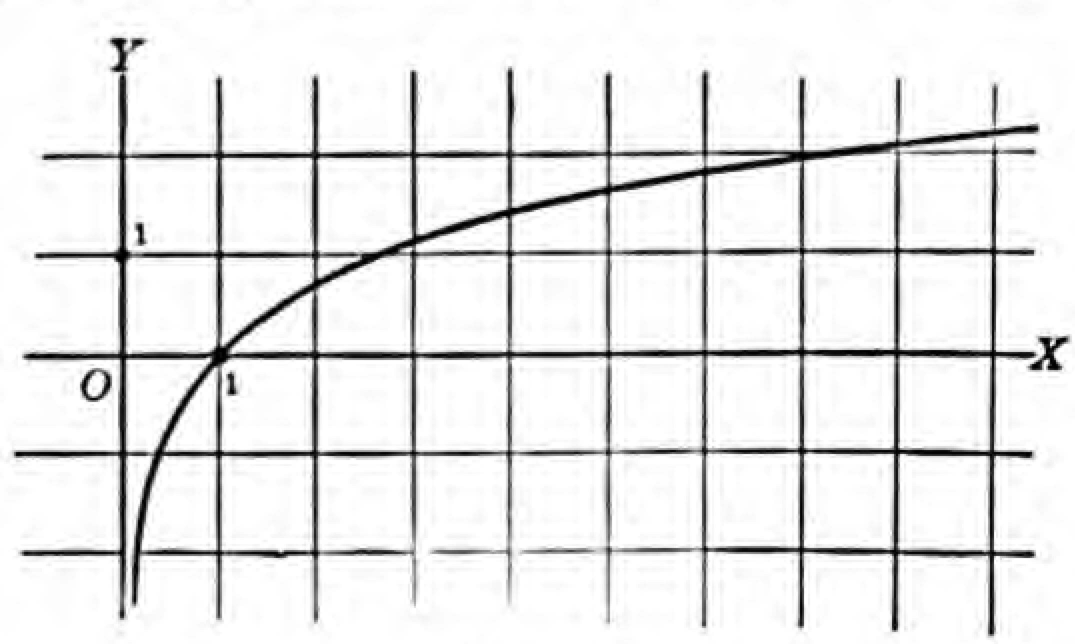


FIG. 27

I. *If x is negative, $\log_a x$ is not defined.*

II. *If $0 < x < 1$, $\log_a x$ is negative, and $\log_a 1 = 0$.*

III. *If x increases without limit, $\log_a x$ increases without limit; if x approaches zero, $\log_a x$ decreases without limit.*

Since $y = \log_a x$ is equivalent to $x = a^y$, *these equations have the same graph.* Thus, in Figure 27 we have a graph of $x = e^y$.

ILLUSTRATION 2. To graph $y = \log_2 x$, we could more easily work with $x = 2^y$; on assigning suitable values to y we could compute the corresponding values of x and then form the graph.

ILLUSTRATION 3. To graph $y = 10^{-\frac{x^2}{4}}$, we would assign values to x and compute y. Thus, if $x = 3$, then $y = 10^{-\frac{9}{4}} = 10^{-2.25}$: hence,

$$\log_{10} y = -2.25 = 7.7500 - 10; \qquad \text{from Table II,} \qquad y = .0056.$$

EXERCISE 77

1. Graph $y = \log_{10} x$ for $0 < x \leqq 30$. From the graph, read the value of $10^{1.3}$; $10^{-.6}$; $10^{.3}$.

2. Graph $y = 2^x$ from $x = -6$ to $x = 4$. From the graph, read $\log_2 6$; $\log_2 10$; $\log_2 .8$.

Graph from $x = -5$ to $x = 5$, using several values of x near to $x = 0$ in the table of values.

3. $y = 10^{-x^2}$. **4.** $y = 10^{-\frac{x^2}{3}}$. **5.** $y = e^{-\frac{x^2}{2}}$. **6.** $y = 3^{-x^2}$.

7. (*a*) Draw a graph of $y = \log_{.5} x$ by graphing $x = .5^y$. (*b*) State facts similar to I, II, and III, above, about $\log_a x$ in case $a < 1$.

★175. A logarithmic scale. In Figure 28, after selecting a unit of length, HK, we lay off a uniform scale on the upper side of the line. On the lower scale, we place each value of x below that value of X which equals $\log x$ (meaning $\log_{10} x$). That is, a value of X, above, and of x, below, satisfy $X = \log x$.

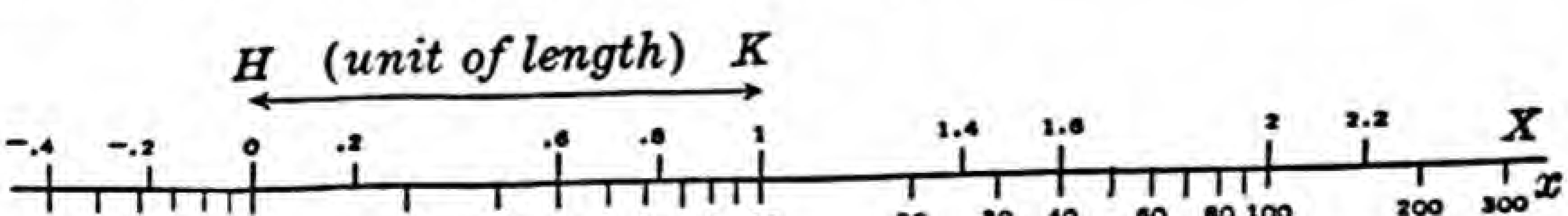


FIG. 28

ILLUSTRATION 1. Since $\log 1 = 0$, hence $x = 1$ is below $X = 0$. Since $\log 40 = 1.6$, approximately, hence $x = 40$ is below $X = 1.6$.

If we blot out the upper scale in Figure 28, we obtain Figure 29, called a *logarithmic scale.* On a log x scale, if the unit of length is properly chosen, *the distance from* $x = 1$ *to* $x = N$ *equals* $\log N$. Since N has no real logarithm if $N \leqq 0$, negative numbers and zero do not appear on a logarithmic scale.

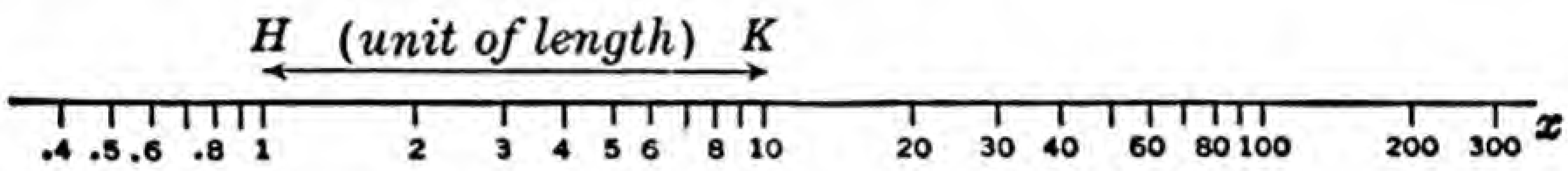


FIG. 29

Note 1. Recall that, if b is any specified base for a system of logarithms, then $\log_{10} N = (\log_{10} b)(\log_b N)$. Hence, if the distance from $x = 1$ to $x = b$ on the scale in Figure 29 is selected as the positive unit of length, then the distance from $x = 1$ to $x = N$ on the scale equals $\log_b N$. Thus, a logarithmic scale made by use of any specified system of logarithms may be used as the logarithmic scale with any other system.

★176. Semilogarithmic coordinates. In Figure 30 there is a logarithmic scale on the vertical axis and a uniform scale on the horizontal axis. To every point P in the plane of these axes, there correspond two numbers (x, y) which we call the *semilogarithmic coordinates* of P. The x-coordinate, or horizontal coordinate, is defined as for rectangular coordinates. The vertical or logarithmic coordinate of P is *the value of* y *on the logarithmic scale at the foot of the perpendicular to it from* P.

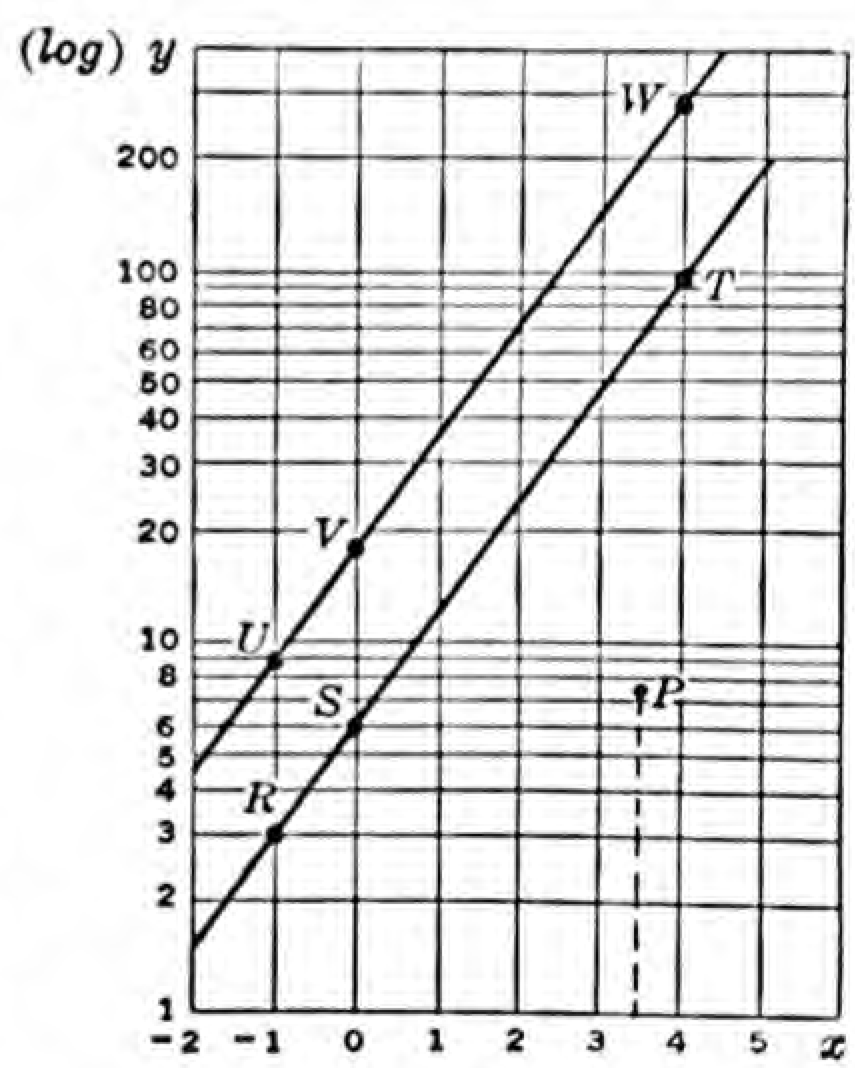


FIG. 30

Now consider the vertical axis in Figure 30 as the Y-axis of an (x, Y) system of rectangular coordinates, where $Y = \log y$; we have $Y = 0$ where $y = 1$, and $Y = 1$ where $y = 10$. Then, to plot a point P whose rectangular coordinates are (x, Y) we plot the point whose *semilogarithmic coordinates* are (x, y); the distance of P from the x-axis is $\log y$.

ILLUSTRATION 1. In Figure 30, the semilogarithmic coordinates of S are $(x = 0, y = 6)$; the *distance* of S above the x-axis is $\log 6$.

ILLUSTRATION 2. To plot P with $(x = 3.5, Y = \log 7.5)$ as the rectangular coordinates, we plot the point whose semilogarithmic coordinates are

(3.5, 7.5). In this act, *we do not need to find* log 7.5 *from a table* because the gradations on the vertical scale show the value of log 7.5.

Note 1. Semilogarithmic coordinates, when used with properly prepared coordinate paper, are designed to aid in graphing points in a rectangular system where one coordinate is most conveniently described as a *logarithm.* The logarithmic scale on the paper makes it unnecessary to employ a table of logarithms in finding the coordinate referred to.

★177. Semilogarithmic graphs. Let (x, y) be a solution of $F(x, y) = 0$, where $y > 0$. Then, with the aid of semilogarithmic coordinate paper we can plot the point whose rectangular coordinates are $(x, \log y)$; the locus of all such points will be called the **$(x, \log y)$-graph** of $F(x, y) = 0$.

EXAMPLE 1. Draw the $(x, \log y)$-graph of $y = 6(2^x)$.

SOLUTION. If $x = 0$, $y = 6$. If $x = -1$, $y = 6(2^{-1}) = 6(\frac{1}{2}) = 3$. If $x = 4$, $y = 96$. In Figure 30, RST is the desired graph:

$$R\colon\ (x = -1, y = 3);\quad S\colon\ (x = 0, y = 6);\quad T\colon\ (x = 4, y = 96).$$

Let $f(x)$ be any *positive* function of x. Then, the locus of all points whose rectangular coordinates are $[x, \log f(x)]$, or whose *semilogarithmic coordinates* are $[x, f(x)]$, will be called the **semilogarithmic graph of $f(x)$.** This graph is the same as the $(x, \log y)$-graph of the equation $y = f(x)$.

ILLUSTRATION 1. In Figure 30, RST is the semilogarithmic graph of the function $6(2^x)$.

THEOREM I. *If a, b, and c are constants, with $a > 0$ and $b > 0$, the semilogarithmic graph of ab^{cx} is a straight line.*

Proof. If we let $y = ab^{cx}$, then $\log y = \log a + cx \log b$. If we let

$$Y = \log y,\quad A = \log a,\quad \textit{and}\quad B = c \log b,$$

then $Y = A + Bx$. Since this equation is *linear* in x and Y, its graph in (x, Y) rectangular coordinates is a *straight line.*

Note 1. When the $(x, \log y)$-graph of $F(x, y) = 0$ is simple in character, we may desire to use it in place of the ordinary graph of the equation.

Note 2. If $a < 0$ and $b > 0$ in $y = ab^x$, then $y < 0$ for all values of x; hence, $y = ab^x$ has no $(x, \log y)$-graph because a negative number has no real logarithm. In such a case, we can let $y = -z$ and thus obtain $z = -ab^x$; then, we may draw the $(x, \log z)$-graph of this equation.

Let $y = f(x)$ and $z = g(x)$ be two positive functions of x, and let the semilogarithmic graphs of $f(x)$ and $g(x)$ be drawn on the same coordinate system. For any value of x, let P be the point on the $(x, \log y)$-graph and Q the point on the $(x, \log z)$-graph. Then, if the x-axis passes through the unit point, 1, on the logarithmic scale, the perpendicular distance

$$\begin{cases} \textit{from the } x\textit{-axis to } P \textit{ is } \log y; \\ \textit{from the } x\textit{-axis to } Q \textit{ is } \log z. \\ \textit{Hence, the distance } QP \textit{ is } (\log y - \log z), \textit{ or } \log \dfrac{y}{z}. \end{cases}$$

Thus, an *increase* in the distance QP as we pass along the graphs indicates an increase in the size of y as compared to z. The semilogarithmic graphs of $f(x)$ and $g(x)$are *parallel curves,* or QP remains *constant,* when and only when $f(x)/g(x)$ *is a constant,* that is, when $f(x)$ *is proportional to* $g(x)$.

ILLUSTRATION 2. In Figure 30, the semilogarithmic graph of $y = 6(2^x)$ is RST and of $z = 18(2^x)$ is UVW. We observe that $z = 3y$, or $3 = \dfrac{z}{y}$, and that the graphs are parallel lines.

★EXERCISE 78*

1. Construct a logarithmic scale, like Figure 29, five inches long, on which x runs from .01 to 100. Use Table II.

2. Plot (0, 10), (5, 30) and (10, 90) on an (x, y) system of rectangular coordinates and also on an $(x, \log y)$ system. Notice that in one case the points fall on a straight line.

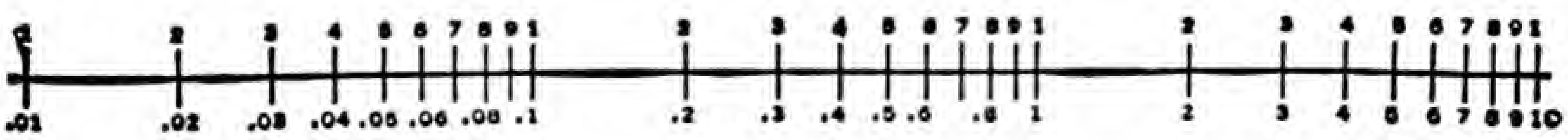
Draw the $(x, \log y)$*-graph, from* $x = -2$ *to* $x = 2$ *if possible on the coordinate paper. In problems* 3 *and* 4, *also draw the ordinary graph.*

3. $y = 10^x$. **4.** $y = e^x$. **5.** $y = 10^{-\frac{1}{2}x}$. **6.** $y = 2(5^x)$. **7.** $y = 6(5^x)$.

8. $y = 3(10^{-.8x})$. **9.** $y = 10^{x^2}$. **10.** $y = 2e^{-x^2}$. **11.** $y = 4e^{-\frac{1}{2}x^2}$.

12. Draw the $(x, \log y)$-graph of $y = 10^{x^2-2x}$ from $x = -1$ to $x = 3$.

* Four-cycle semilog paper is desirable. On a log scale on the paper, the "1" at the lower (or left) end of the scale may be marked as any convenient power of 10, and the other numbers on the scale are then altered correspondingly. Thus, if the log scale appears on the coordinate paper as shown by the upper numbers on the following scale, we might alter them as given below the line:



13. In the formula of compound interest, $F = P(1 + r)^k$, let $P = 100$ and $r = .04$ and draw the $(k, \log F)$-graph.

14. In Problem 19, page 184, let $y = I/A$ and draw the $(t, \log y)$-graph. Then, use this graph to solve Problem 19 graphically.

15. What fact is true about $f(x)$ and $g(x)$ if there is a constant vertical distance between corresponding points on their graphs on the same system of *rectangular* coordinates? (Compare with Section 177.)

16. The table shows the world production, w, and the United States production, y, of petroleum in 1,000,000's of barrels for various years. Plot the $(t, \log y)$- and the $(t, \log w)$-graphs on *one* semilogarithmic system. From the graphs, read various facts about the ratio $y:w$.

y	356	472	714	901	906	997	1265	1353
w	503	766	1014	1252	1442	1643	2085	2142
YEAR (t)	1918	'21	'24	'27	'33	'35	'39	'40

17. The table gives the United States production (y) of cigarettes in billions for various years. Plot the $(t, \log y)$-graph and, from it, estimate the production in 1930 and 1944.

YEAR (t)	1934	'35	'36	'37	'38	'39	'40	'41	'42
y	130	140	159	170	172	181	189	218	258

★178. Logarithmic coordinates. In Figure 31, each axis is supplied with a logarithmic scale. To each point P there correspond two numbers (x, y) which we call the *logarithmic coordinates*, or,

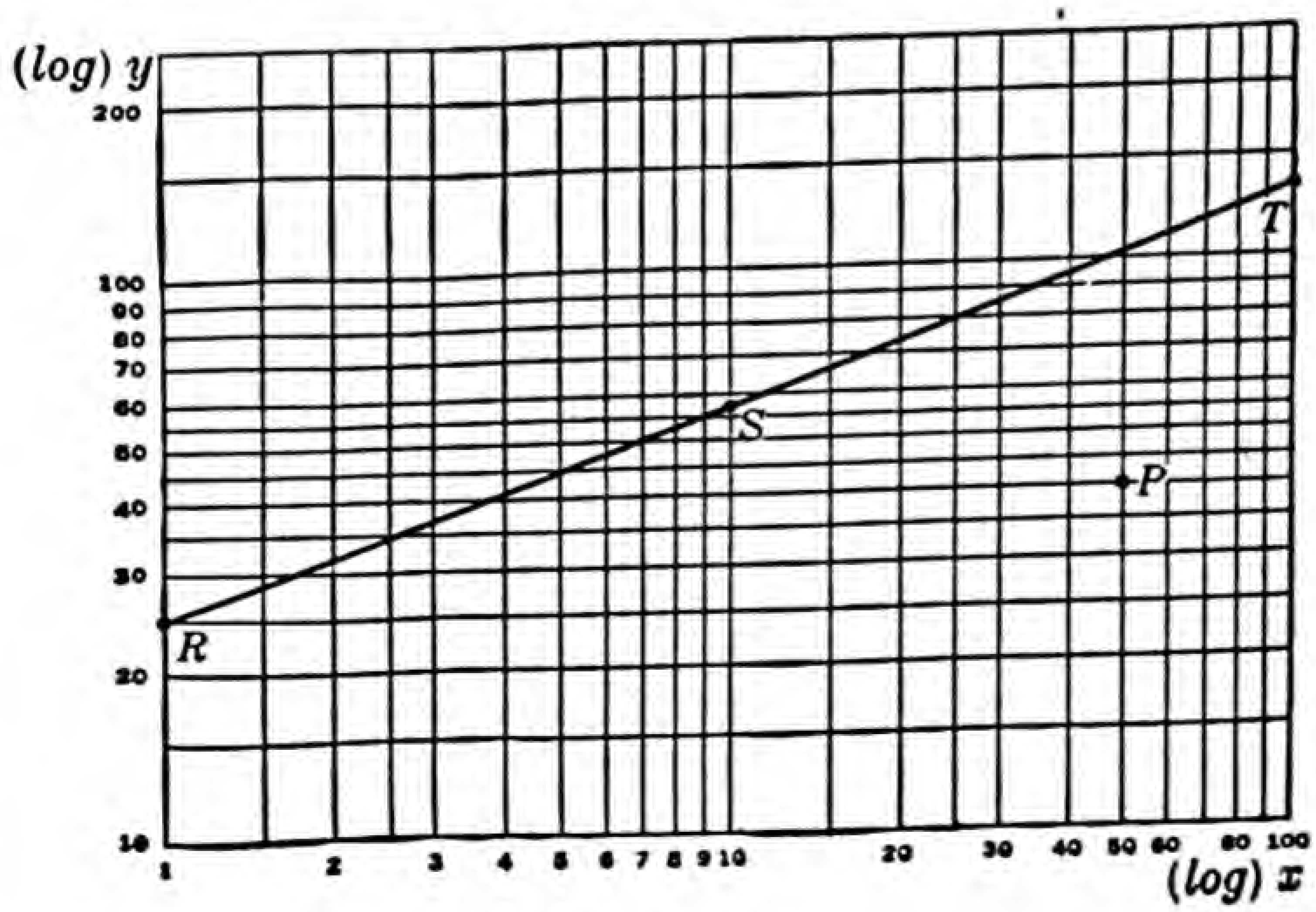


FIG. 31

for short, the *log-log* coordinates of P. The log-log coordinates of P are the values of x and y on the axes at the feet of the perpendiculars to the axes from P.

Let us also consider the axes in Figure 31 as those of an (X, Y) system of *rectangular* coordinates with the origin at $(x = 1, y = 1)$ and with $(X = \log x, Y = \log y)$. To plot the point whose *rectangular* coordinates are $(\log x, \log y)$, we plot the point whose *log-log* coordinates are (x, y).

ILLUSTRATION 1. In Figure 31, the log-log coordinates of P are $(50, 40)$ and the related rectangular coordinates are $(\log 50, \log 40)$.

The locus of all points whose log-log coordinates (x, y) form a solution of $F(x, y) = 0$ will be called the **(log x, log y)-graph** of $F(x, y) = 0$. The locus of all points whose log-log coordinates are $[x, f(x)]$ will be called the **logarithmic graph** of the function $f(x)$; this graph is the same as the $(\log x, \log y)$-graph of the equation $y = f(x)$.

Note 1. Since N has no logarithm if $N \leqq 0$, both x and y must be positive in a $(\log x, \log y)$-graph.

EXAMPLE 1. Obtain the $(\log x, \log y)$-graph of $y = 25x^{.3602}$.

SOLUTION. If $x = 1$, then $y = 25$. If $x = 10$, then $y = 25(10^{.3602})$ and

$$\log y = \log 25 + .3602 = 1.7581;$$

hence
$$y = \text{antilog } 1.7581 = 57.$$

If $x = 100$, we obtain $y = 131$. In Figure 31 we plot the points

R: $(x = 1, y = 25)$; S: $(x = 10, y = 57)$; T: $(x = 100, y = 131)$.

The straight line RST is the desired $(\log x, \log y)$-graph.

THEOREM I. *If a and b are constants and if a is positive, then the* $(\log x, \log y)$*-graph of* $y = ax^b$ *is a straight line.*

Note 2. The student should prove Theorem I.

★EXERCISE 79

Use log-log coordinate paper.

1. Plot $(1, 2)$, $(2, 16)$, and $(4, 128)$ on an (x, y) system of rectangular coordinates. Also, plot these pairs (x, y) on a $(\log x, \log y)$ system. Notice that in one case the points lie on a straight line.

Plot the $(\log x, \log y)$*-graph of each equation.*

2. $y = x^2$. **3.** $y = 5x^3$. **4.** $y = 10x^{-2}$. **5.** $y = 2x^{-3}$. **6.** $y = 5(x^{1.635})$.

CHAPTER FOURTEEN

Permutations and Combinations

179. Fundamental principle. *If one act can be performed in h different ways and if, after its performance, a second act can be performed in k different ways, then the two acts can be done in the stated order in hk different ways.*

Proof. For each way* of doing the first act there are k ways of doing the two acts in the stated order. Hence, since there are h ways of doing the first act, there are hk ways of doing the two acts in the stated order.

ILLUSTRATION 1. If there are 5 ways of going from A to B and 4 ways of going from B to C, then we can go from A to B to C in $5 \cdot 4$ or 20 ways.

Two or more acts are said to be *independent* if the performance of any one does not affect the performance of the other acts. The *order* of the performance of *independent* acts is immaterial and hence we can restate the fundamental principle as follows: *if one act can be performed in h ways and a second independent act in k ways, then the two can be done together in hk ways.*

An evident extension of the principle holds for three or more acts.

Note 1. In this chapter, unless otherwise specified, the word *number* will mean a *positive integer*.

EXAMPLE 1. How many numbers of three different digits each can be formed from the digits 1, 2, 3, 5, 8, and 9?

SOLUTION. We can choose any one of the six digits for the units' place, then any one of the five remaining digits for the tens' place, and then any one of the four remaining digits for the hundreds' place. Hence, we can form $6 \cdot 5 \cdot 4$ or 120 different numbers of the specified kind.

$\boxed{(4)}$ $\boxed{(5)}$ $\boxed{(6)}$

EXAMPLE 2. How many numbers of four different digits each can be formed from the digits 1, 2, 3, 4, 5, 6, and 8, if each number is to begin and end with an odd digit?

* In referring to "ways" we shall mean *different* ways.

SOLUTION. We can choose any one of the three digits (1, 3, 5) for the thousands' place, then any one of the two remaining odd digits for the units' place, then any one of the five remaining digits for the hundreds' place, etc. Hence, we can form $3 \cdot 5 \cdot 4 \cdot 2$ or 120 different numbers of the specified kind.

$$\boxed{(3)} \quad \boxed{(5)} \quad \boxed{(4)} \quad \boxed{(2)}$$

EXAMPLE 3. In how many ways can 3 men be assigned consecutive seats in a row of 7 seats?

SOLUTION. 1. We can select 3 consecutive seats in any one of 5 ways (think first of the three being at the left end, and then move from this position to the right end one seat at a time, in 4 steps).

2. After any choice of the 3 consecutive seats, the 3 men can be assigned places in these seats in $3 \cdot 2 \cdot 1$ or 6 ways.

3. By the fundamental principle, Steps 1 and 2 can be performed in $5 \cdot 6$ ways. Hence, the men can be seated in 30 ways.

180. Permutations. We can think of any set of things as being arranged in a set of numbered places; for instance, in places numbered 1, 2, 3, $\cdots$. Any such ordered arrangement of any part of a set of things is called a *permutation* of them. If r of the things occur in the arrangement, it is called a permutation of the things *taken r at a time.*

ILLUSTRATION 1. The permutations of the letters a, b, and c, taken two at a time, are ab, ba, ac, ca, bc, and cb; their permutations, taken three at a time, are abc, acb, bac, bca, cab, and cba.

EXAMPLE 1. Find the number of permutations of 7 different things taken 3 at a time.

SOLUTION. In forming permutations, we can fill the first place in 7 ways, then the second place with any one of the 6 things remaining after the first place is filled, and finally the last place in 5 ways. Hence, there are $7 \cdot 6 \cdot 5$ or 210 permutations of the specified kind.

EXERCISE 80

1. How many numbers of four different digits each can be formed from the digits 2, 5, 6, 7, 1, and 8?

2. In how many ways can 3 positions be filled by selections from 15 people?

3. If 5 coins are tossed together, in how many ways can they fall?

4. A girl has invited 6 friends to a dinner party. After locating herself at the table, in how many ways can she arrange her friends in the remaining seats?

5. There are 4 applicants for 3 different positions. In how many ways can the positions be filled?

6. Two cubical dice are tossed together. In how many ways can they fall?

7. How many permutations are there of the letters x, y, z, and w, taken three at a time? Write out the permutations.

8. How many permutations are there of the digits 1, 2, 3, 4, and 5, taken three at a time? Write out these permutations.

9. How many permutations are there of the letters in the word *hotel*, (*a*) taken five at a time; (*b*) taken three at a time?

10. Find the number of permutations of n different things taken three at a time.

11. In how many ways can a teacher seat 6 pupils if 8 seats are available?

12. In how many ways can 3 men take up quarters in a city where there are 5 hotels available?

13. In how many ways can a man distribute a nickel, a dime, and a quarter among 6 boys?

14. From the digits 1, 2, 3, 4, 5, 6, 7, 8, and 9, we shall form all possible numbers of four different digits each. (*a*) How many numbers do we form? (*b*) How many of these numbers are divisible by 5? (*c*) How many are odd? (*d*) How many both begin and end with an odd digit?

15. From the letters of the word *fancies*, we shall form all possible permutations of 5 letters each. (1) How many permutations do we form? (2) How many begin and end with a vowel? (3) How many have *s* or *c* in the middle? (4) How many end with a consonant?

16. How many distinct license plates for automobiles can be made up if each plate label consists of a capital Roman letter followed by a number, not zero, less than 1,000,000?

17. How many numbers of five different digits each can be formed from the digits 0, 2, 3, 4, 5, and 7?

18. How many numbers greater than 4000, of four different digits each, can be formed from the digits 0, 3, 6, 7, 4, and 2?

19. In how many ways can 7 different books be placed in a row on a shelf in case neither of 2 particular books is to be at either end?

20. How many numbers can be formed by use of the digits 1, 2, 3, and 5, if no digit is used more than once in a number?

21. In how many ways can a teacher seat 4 boys and 3 girls in a row of 7 seats, if the boys and girls are to alternate?

22. In how many ways can 2 people take places in consecutive seats in a row of 8 seats?

23. In how many ways can a certain 4 men be assigned positions in the batting order for a baseball nine, if they are to occupy consecutive positions?

24. How many numbers of five different digits each can be formed from the digits 1, 2, 3, 4, 5, 6, and 7, if odd digits are to stand at each end and in the middle?

25. In how many ways can 3 individuals be assigned consecutive positions in a receiving line of 12 people?

26. How many numbers greater than 53,000 can be made from the digits 2, 3, 4, 5, 6, and 7, without repeating digits in any number?

27. A motor bus has 5 seats vacant on each side. In how many ways can 7 persons A, B, C, D, E, F, and G be seated, with A, B, and C on the right and the others on the left side?

28. In how many ways can 5 people be seated in a row of 5 seats if a certain 2 people are not to sit at either end?

29. By use of (1, 2, 3, 4, 5, 6), how many numbers of five different digits each can be formed in which odd and even digits alternate?

181. Formulas for permutations of different things. Let the symbol ${}_nP_r$ represent the number of permutations of n different things taken r at a time.

ILLUSTRATION 1. We read "${}_5P_3$" as the number of permutations of five things taken three at a time.

THEOREM I. *The number of permutations of n different things taken r at a time is* $n(n-1)(n-2)\cdots(n-r+1)$:

$$ {}_nP_r = n(n-1)(n-2)\cdots(n-r+1). \tag{1}$$

Proof. In any permutation, we can fill the first place by any of the n things, then the second place by any of the $(n-1)$ things remaining after the first place is filled, then the third place by any of the $(n-2)$ things remaining, $\cdots$, finally the rth place by any of the $[n-(r-1)]$ things remaining after the $(r-1)$th place is filled. Hence, by the fundamental principle, all r places in a permutation of the n things, taken r at a time, can be filled in

$$n(n-1)(n-2)\cdots(n-r+1) \textit{ different ways.}$$

COROLLARY 1. *The number of permutations of n different things, taken n at a time, is $n\,!$.*

Proof. We place $r = n$ in (1) and obtain

$$P_n = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1 = n\,!. \tag{2}$$

ILLUSTRATION 2. ${}_7P_4 = 7\cdot 6\cdot 5\cdot 4 = 840$; ${}_nP_3 = n(n-1)(n-2)$.

Note 1. Whenever convenient, the student should find the number of permutations by direct application of the fundamental principle, as in the preceding exercise. However, after the principle is thoroughly appreciated, it is frequently desirable to employ the formulas which we have just derived, particularly formula 2.

EXAMPLE 1. In how many different orders can 7 people take seats at a round table?

SOLUTION. The positions of any six may be thought of with reference to the seventh person, whom we may consider fixed in position. Hence, the seven can be seated in ${}_6P_6$, or $6\,!$ different orders.

Note 2. In the preceding example, we dealt with **cyclical permutations** (arrangements in a *ring*). In contrast, ordinary permutations can be thought of as arrangements in a *straight line*, or **linear permutations.** The number of *cyclical* permutations in Example 1 is $6\,!$, whereas the number of *linear* permutations of 7 people, taken 7 at a time, is $7\,!$.

182. Permutations of things not all different. It is easily seen that there are fewer permutations of *like* things than of *unlike* things. For instance, there are 6 permutations of (a, b, c) taken all at a time, whereas the only permutation of the three letters (a, a, a), taken all at a time, is aaa.

EXAMPLE 1. Find the number of permutations of (a, a, a, b, c) taken five at a time.

SOLUTION. 1. Let P be the desired number of permutations.

2. Consider (a_1, a_2, a_3, b, c) where all letters are different; their number of permutations taken five at a time is $5\,!$. We can obtain these as follows:

I. *take in turn each distinct permutation of* (a, a, a, b, c), *and*

II. *replace the* a*'s in all possible ways by* (a_1, a_2, a_3).

We can perform (I) in P ways, (II) in $3\,!$ ways, and hence (I) and (II) in $P(3\,!)$ ways. Therefore,

$$P(3\,!) = 5\,!, \quad \text{or} \quad P = \frac{5\,!}{3\,!} = 20.$$

THEOREM I. *If P represents the number of distinct permutations of n things taken all at a time when, of the n things, there are u alike, v others alike, w others alike, etc., then*

$$P = \frac{n\,!}{u\,!\,v\,!\,w\,!\cdots}. \tag{1}$$

Proof. 1. For concreteness, let the n things consist of u like things represented by a's, and v other like things represented by b's.

2. Now, replace the u letters "a" by u different letters $a_1, a_2, \cdots, a_u$ and the v letters "b" by v different letters $b_1, b_2, \cdots, b_v$, thus obtaining n *different* letters. We can create all their permutations taken n at a time as follows:

I. *Take in turn each permutation of the original u letters "a" and v letters "b."*

II. *Then replace the u letters "a" of the permutation in all possible ways by the u different letters $a_1, a_2, \cdots, a_u$.*

III. *Replace the v letters "b" in all possible ways by the v different letters $b_1, b_2, \cdots, b_v$.*

We can do (I) in P ways, (II) in $u\,!$ ways, and (III) in $v\,!$ ways. Hence, we can do (I), (II), and (III) in succession in $P(u\,!)(v\,!)$ ways. But, this equals the number of permutations of the n different letters taken n at a time, or $n\,!$. Hence,

$$P(u\,!)(v\,!) = n\,!, \quad or \quad P = \frac{n\,!}{u\,!\,v\,!}. \tag{2}$$

ILLUSTRATION 1. The number of permutations of the letters in *attention* taken all at a time is $\dfrac{9\,!}{3\,!\,2\,!}$, because there are three t's and two n's.

EXERCISE 81

Read each symbol and compute its value.

1. ${}_5P_2$. **2.** ${}_6P_4$. **3.** ${}_3P_3$. **4.** ${}_7P_3$. **5.** ${}_{12}P_3$. **6.** ${}_{21}P_2$.

Find the number of distinct permutations of the letters or digits, taken all at a time.

7. $(a, a, a, b, b, b, c, c, c, c)$. **8.** $(2, 2, 2, 3, 4, 4, 4, 5, 5)$.

9. How many distinct permutations can be made of the letters of the word *attention*, taken all at a time?

10. How many different numbers of eight digits each can be formed by use of two 2's, two 3's, three 4's, and one 5, in each number?

11. How many different numbers of six digits each can be formed by using the digit 5 twice, 4 three times, and 7 once?

12. In how many ways can 4 dimes and 6 quarters be distributed among 10 boys, if each is to receive one coin?

13. In signaling with 7 similar flags, they will be arranged in order from left to right. How many different signals can be made involving all the flags, if *three* are red, *two* are white, *one* is blue, and *one* is green?

14. (*a*) In how many different orders can 9 people be seated at a round table? (*b*) In how many ways can they be arranged in a line facing north?

15. In how many ways can 8 boys (*a*) form a ring facing its center; (*b*) be seated in a row of 8 chairs?

16. From the letters A, B, C, D, E, F, and G, we shall form all permutations, taken five at a time. (*a*) How many do we form? (*b*) In how many will B not be present? (*c*) In how many will A be found at one or other of the ends? (*d*) In how many will C be present?

17. From the digits 1, 2, 3, 5, 7, and 9, we shall form all possible numbers of four different digits each. (*a*) How many numbers will be formed? (*b*) In how many of the numbers will the digit 1 appear in the thousands' place? (*c*) How many will not involve the digit 5? (*d*) How many will have 7 at one or other of the ends?

18. In how many essentially different ways can 8 beads of different colors be arranged to form a necklace?

19. How many numbers of five different digits each can be formed by use of (1, 2, 3, 4, 5) with 3 and 4 consecutive?

HINT. We analyze the formation of any such number into three stages: (*a*) we choose two places in the number for (3, 4); (*b*) we arrange (3, 4) in these two places; (*c*) we arrange the other three digits in their places. (*a*) can be done in 4 ways, (*b*) in 2 ways, etc. Then use the fundamental principle of Section 179, applied to the successive actions (*a*), (*b*), and (*c*).

20. In how many ways can 7 people take seats in a row of 7 seats, if three particular people insist on being side by side?

21. In how many ways can we form permutations of (a, b, c, d, e, f, g), taken all at a time, if b and g are to be side by side?

22. In how many essentially distinct ways can 7 different keys be assembled on a key ring?

23. In how many ways can the manager of a baseball team arrange his 9 men in the batting order if he wishes the 4 best hitters to be in the first 4 positions?

24. In how many ways can 4 different books on history and 3 different books on mathematics be arranged on a shelf, if all books on the same subject are to be side by side?

25. In how many ways can a red book, a green book, and 5 different blue books be arranged together on a shelf (*a*) with the red and green books separated; (*b*) with the red and green books consecutive?

26. How many numbers of five different digits each can we form from (1, 3, 4, 5, 7), if 3 and 4 are *not* to be consecutive?

27. (*a*) In how many ways can 5 men and 5 women be formed into couples in the first 5 positions in a grand march? (*b*) In how many orders can they be arranged in a circle with men and women alternating and facing the center?

28. In how many ways can 6 different books be arranged on a shelf with books of the same color side by side, if 2 books are red and 4 are blue?

29. In how many distinct orders can 4 different plates and 5 larger identical plates be arranged at the 9 places of a round table?

30. In how many distinct ways can 3 identical keys and 4 smaller different keys be assembled on a key ring?

31. In how many ways can we assign 11 men to positions on a football team if only 3 men are qualified to play at the ends and only 2 other men can play at quarterback, but if all men can play at the other 8 positions?

183. A combination of a set of things is a group of all or of any part of the things, **without regard to the order of the things in this group.** A combination involving r of the things is called a combination of the things *taken r at a time.*

ILLUSTRATION 1. The different combinations of a, b, c, and d, taken 3 at a time, are (a, b, c), (a, b, d), (a, c, d), and (b, c, d). From each combination we can form $3!$ or 6 different permutations of the 4 letters taken 3 at a time. Thus, from (a, b, c) we can form the permutations *abc, acb, bac, bca, cab,* and *cba.* In other words, there are only *four combinations,* whereas there are $4 \cdot 6$, or 24, *permutations* of the 4 letters taken 3 at a time.

We use the symbol ${}_nC_r$ to denote the number of combinations of n different things taken r at a time.

ILLUSTRATION 2. ${}_7C_3$ represents the number of combinations of 7 different things taken 3 at a time.

THEOREM I. *The number of combinations of n different things taken r at a time equals the number of permutations of n different things, taken r at a time, divided by r!.*

Proof. With each combination containing r of the things, we can form $r!$ permutations of the things taken r at a time. Hence, since there are ${}_nC_r$ different combinations, there are ${}_nC_r \cdot (r!)$ different permutations. That is, ${}_nC_r \cdot (r!) = {}_nP_r$, or

$$ {}_nC_r = \frac{{}_nP_r}{r!}. \tag{1} $$

Since ${}_nP_r = n(n-1)(n-2)\cdots(n-r+1)$,

$$ {}_nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r\,!}. \tag{2}$$

If both numerator and denominator in (2) are multiplied by $(n-r)!$, the new numerator is

$$1\cdot 2\cdot 3\cdots(n-r)(n-r+1)\cdots(n-2)(n-1)n,$$

or $n\,!$, and therefore

$$ {}_nC_r = \frac{n!}{r\,!(n-r)!}. \tag{3}$$

Note 1. Formula 1 is recommended for use in computing ${}_nC_r$. For later use, formula 3 is recommended instead of formula 2.

Example 1. From 10 people, in how many ways can we (*a*) select a *group* * of 6 people; (*b*) fill 6 different offices in a club?

Solution. (*a*) Since no order is assigned in the group, the result is the number of *combinations* of 10 people taken 6 at a time, which is ${}_{10}C_6$ or 210.

(*b*) The result is ${}_{10}P_6$ or 151,200, the number of *permutations* of 10 people taken 6 at a time.

Whenever we pick a group of r things from n things, we leave, or set aside, $(n-r)$ of the things. Hence, the number of combinations of n different things, taken r at a time, is the same as the number of combinations of n different things, taken $(n-r)$ at a time, or

$$ {}_nC_r = {}_nC_{n-r}. \tag{4}$$

★*Note 2.* Formula 4 can also be proved by use of (3). From (3),

$$ {}_nC_{n-r} = \frac{n\,!}{(n-r)!\,[n-(n-r)]\,!} = \frac{n\,!}{(n-r)!\,r\,!} = {}_nC_r.$$

Illustration 3. ${}_{50}C_{48} = {}_{50}C_2 = \dfrac{50.49}{2} = 1225.$

184. Mutually exclusive events. If a certain two events cannot occur at the same time, we shall call them *mutually exclusive.* For such events, we observe the following simple principle.

If a first event can occur in h ways and a second event in k ways, and if the events are mutually exclusive, then one or the other of them can occur in $h+k$ ways.

* We shall always infer that, in a *group,* no *order* is assigned to the individuals.

ILLUSTRATION 1. Suppose that a bag contains 5 white, 6 black, and 4 red balls. By the preceding principle, we observe that we can draw a white ball *or* a black ball from the bag in $(5 + 6)$ or 11 ways. On the other hand, by the principle of Section 179 for successive events, we can draw a white ball *and* a black ball in $5 \cdot 6$ or 30 ways. The key words "OR" and "AND" gave the essential clues leading, respectively, to the principles of Sections 184 and 179.

EXAMPLE 1. From 6 men and 5 women, in how many ways can we select a group (*a*) of 4 men *or* of 3 women; (*b*) of 4 men *and* 3 women?

SOLUTION. 1. We can select a group of 4 men in ${}_6C_4$ or 15 ways.

2. We can select a group of 3 women in ${}_5C_3$ or 10 ways.

3. (*a*) By the principle for *mutually exclusive events*, we can select a group of 4 men OR 3 women, that is, perform Step 1 OR Step 2, in $(15 + 10)$ or 25 ways.

4. (*b*) By the fundamental principle of Section 179 for *successive events*, we can select a group of 4 men AND 3 women, that is, perform Step 1 AND Step 2 in succession, in $15 \cdot 10$ or 150 ways.

EXERCISE 82

1. (*a*) How many combinations are there of (E, H, K, M), taken three at a time? (*b*) Write out all of these combinations. (*c*) How many permutations of the 4 letters, taken 3 at a time, can be formed with each of the preceding combinations? (*d*) Write out all the permutations you can form from the combination (E, H, M).

Read each symbol in words, and compute it.

2. ${}_8C_3$. **3.** ${}_7C_5$. **4.** ${}_{12}C_5$. **5.** ${}_6C_3$.

6. From a group of 10 friends, in how many ways can a man select a dinner party of 8 people?

7. How many sums of money, each composed of 3 coins, can be made from a dime, a dollar, a quarter, and a cent?

8. (*a*) How many different straight lines can be drawn through points selected from 12 points in a plane, if no three of these points are in a line? (*b*) How many of these lines will go through any one point?

9. From (c, d, e, f, g, h, k), how many combinations of 4 letters each can be formed including the letter d?

10. From a club of 10 people, (*a*) how many different committees* of 4 people each could be formed; (*b*) in how many ways could we fill the positions of president, vice-president, secretary, and treasurer of the club?

* We agree that no *order* of precedence is designated on a committee.

11. A bag contains 8 black, 7 white, and 5 green balls.* In how many ways can we select groups of balls consisting (*a*) of 4 black *or* 4 white balls; (*b*) of 4 black *and* 4 white balls; (*c*) of 4 balls all of the same color?

12. Use the formula ${}_nC_r = {}_nC_{n-r}$ to compute ${}_{1000}C_{998}$.

13. How many different groups of 1500 people each can be formed from 1502 people?

14. From a suit of 13 playing cards, how many hands of 5 cards each can be dealt to a player?

15. In how many ways can we select groups of 8 books, consisting of 5 English books and 3 German books, from 10 different English books and 6 different German books?

16. From 8 men and 6 women, how many committees of 5 each can be formed if each committee consists of (*a*) 3 men and 2 women; (*b*) entirely of men or entirely of women?

185. Miscellaneous methods. In determining the number of ways of performing a complicated act, it is frequently useful to make a preliminary analysis of the act into either *successive* simpler acts, or into various *mutually exclusive* simpler acts. After the numbers of ways of performing the simpler acts have been found, the principles of Sections 179 and 184 can be employed to obtain the final result.

EXAMPLE 1. How many numbers of five different digits each can be formed if each number involves three odd and two even digits and no digit 0?

SOLUTION. 1. In forming a number, we must

(*a*) *select a group of three odd digits from* (1, 3, 5, 7, 9),

(*b*) *select a group of two even digits from* (2, 4, 6, 8), *and then*

(*c*) *form a permutation of the five digits selected in* (*a*) *and* (*b*).

2. We can do (*a*) in ${}_5C_3$ or 10 ways, and (*b*) in ${}_4C_2$ or 6 ways. For each way of selecting three odd and two even digits, we can do (*c*) in ${}_5P_5$ of $5!$ ways. Hence, by the fundamental principle of Section 179, we can do (*a*), (*b*), and (*c*) in succession in $10 \cdot 6 \cdot 5!$ or 7200 ways.

EXAMPLE 2. How many numbers without repeated digits and greater than 43,000 can be made by use of (2, 3, 4, 5, 6, 7)?

PARTIAL SOLUTION. The number may have (*a*) six digits, or (*b*) five digits starting with 4, or (*c*) five digits starting with 5, 6, or 7. Types *a*, *b*, and *c* are *mutually exclusive*. The student should find how many numbers exist of type *a*, of type *b*, and of type *c*, and then add the results, in accordance with Section 184. The final result is 1176.

* Indistinguishable except as to color. This assumption will persist in all problems in this book with similar data.

EXAMPLE 3. A bag contains 7 black and 6 white balls. In how many ways can we draw from the bag groups of 5 balls involving *at least* 3 black balls?

SOLUTION. 1. We obtain *at least* 3 black balls if we obtain

I. EXACTLY 3 black and 2 white balls; OR

II. EXACTLY 4 black and 1 white ball; OR

III. EXACTLY 5 black balls.

2. Since (I), (II), and (III) are *mutually exclusive events*, we *add* their numbers of ways of occurrence and obtain the following final results:

$$({}_7C_3)({}_6C_2) + ({}_7C_4)({}_6C_1) + {}_7C_5 = 756 \text{ ways.}$$

Comment. Notice the preliminary analysis of "AT LEAST" into various mutually exclusive possibilities involving "EXACT" situations, to which our methods apply more easily. A similar analysis is usually advisable in any problem involving "AT MOST" in its statement.

MISCELLANEOUS EXERCISE 83

1. From a group of 6 different books, in how many ways can we choose groups (*a*) of exactly 3 books each; (*b*) of 3 or more books each?

2. From a penny, a nickel, a dime, a quarter, and a half dollar, in how many ways can we form sums (*a*) of exactly 2 coins each; (*b*) of 2 or more coins each?

3. In how many ways can 4 different presents be distributed among 3 children?

4. From the digits (1, 2, 3, 4, 5, 6, 7, 8, 9), how many numbers of four different digits each can be formed, each number consisting of two odd and two even digits?

5. From the letters (a, e, i, s, t, v, w), how many permutations of 5 letters each can be formed, each permutation consisting of 3 consonants and 2 vowels?

6. In how many ways can 5 people take seats in a row of 8 seats?

7. In how many ways can 6 boys choose places in 9 different seats?

8. From a group consisting of 7 men and 5 women, in how many ways can we select (*a*) a group consisting of 3 men and 2 women; (*b*) a group of 5 people containing at least 3 men?

9. From a bag containing 6 black and 8 white balls, in how many ways can we form a group of 5 balls containing (*a*) exactly 2 black balls; (*b*) at most 2 black balls?

10. In a baseball league of 9 teams, how many games will be played if each team plays 12 games with each other team?

11. In how many ways can a minstrel troupe of 9 men be seated in a row on the stage, if 2 particular men must act as end men?

12. In how many ways can 8 different presents be divided between A, B, and C so that A will receive three, B three, and C two presents?

HINT. A can be given presents in ${}_8C_3$ ways; then, five remain.

13. In how many ways can a group of 12 people be divided between 3 automobiles, four in one, five in another, and three in the third automobile?

14. From a set of 15 different books, in how many ways can A, B, and C be given 4, 6, and 3 books, respectively?

15. From a group of 6 freshmen, 5 sophomores, and 8 seniors, how many committees can be formed if each consists of 3 members of one class and 3 of another class?

16. From the letters (a, b, c, d, e, f), how many permutations of 6 letters each can be formed in which (a, b, c, f), in some order, stand in the first 4 places and no letter is repeated in the permutation?

17. From the digits (1, 2, 3, 4, 5, 6, 7, 8), how many numbers of six different digits each can be formed in which odd and even digits alternate?

18. One bag contains 5 white and 6 black balls, and a second bag contains 4 white and 7 black balls. In how many ways can we select a group of 5 balls, consisting of 3 white and 2 black balls, (*a*) if all balls must come from the same bag; (*b*) if the white ones come from one bag and the black ones from the other bag?

19. How many different committees of 5 men each can be selected from 9 men, if a certain 2 men refuse to serve together on any committee?

20. If ${}_nP_4 = 840$, find ${}_nC_4$.

21. If ${}_nC_6 = 210$, find ${}_nP_6$.

22. If 5 coins are tossed together, in how many ways can it result (*a*) that all fall heads; (*b*) that two fall heads and three fall tails; (*c*) that at least three fall tails?

23. A man will play a certain game 7 times, and we can arrange for him either to win or to lose in any game. In how many ways can we arrange that he should win (*a*) exactly 5 of the 7 games; (*b*) at most 3 games?

24. In how many ways can 6 different red books and 4 different black books be exhibited on a shelf, if each red book has a red book on one side and a black book on the other side?

25. In how many ways can 6 people be seated in a row of 6 seats, if a certain 2 people are not to be side by side?

26. In how many different orders can we seat 8 people at a round table if a certain 2 people (*a*) are to sit next to each other; (*b*) are not to sit next to each other?

27. A teacher has 8 different assignments for outside work to give to a group of 4 students. In how many ways can the assignments be distributed, two to each student?

28. From the letters of the word *amicable*, how many permutations of six letters each can we form in which the first four letters are vowels?

29. If 7 coins are tossed together, in how many ways can they fall with (*a*) just 4 heads; (*b*) at least 4 heads?

30. If the faces of a cubical die are numbered 1, 2, 3, 4, 5, and 6, respectively, and if 2 dice are tossed, in how many ways can a total of 6 show up on the dice?

31. How many numbers without repeated digits and greater than 3,200 can be made by use of (1, 2, 3, 0, 5)?

32. How many hands each consisting of 6 hearts and 7 spades can be made up from a usual deck of cards?

33. A box contains 7 different red books and 5 different blue ones. In how many ways can we exhibit six of these books in a row on a shelf, if each exhibit should contain 4 red and 2 blue books?

34. Find the number of distinguishable combinations of the letters (a, a, b, c, d, e, f, g, h), taken three at a time.

35. A motor bus has 5 seats on each side. In how many ways can 6 people seat themselves, if a certain two always take seats on the right side and a third always sits on the left side?

36. (*a*) How many different hands of 13 cards each can be made from a deck of 52 cards? (*b*) In how many ways can 4 people in a game be dealt hands of thirteen each? Leave the results factored.

37. In how many ways can 9 different books be exhibited on a shelf if 4 books are red, 3 are white, and 2 are green, and if books of the same color stand side by side?

38. In how many different orders can 4 men and 4 women be seated at a round table with men and women in alternate seats?

39. How many numbers of five digits each can be formed by use of (1, 2, 3, 4, 5) if each number involves two 3's but no other repeated digits?

40. In how many ways can a hostess seat herself and 9 guests at a round table, with a certain 3 guests consecutive and with 2 others consecutive?

Solve each equation for n.

41. ${}_nP_3 = 336$. **42.** ${}_nP_7 = {}_nP_6$. **43.** ${}_nC_4 = 210$.

★186. Binomial coefficients as combination symbols. On comparing (2) on page 200 with (2) on page 101, we observe that, in the expansion of $(x + y)^n$,

$$\text{the term involving } y^r \text{ is } {}_nC_rx^{n-r}y^r: \tag{1}$$

$$(x + y)^n = x^n + {}_nC_1x^{n-1}y + {}_nC_2x^{n-2}y^2 + \cdots + {}_nC_rx^{n-r}y^r + \cdots + {}_nC_ny^n. \tag{2}$$

ILLUSTRATION 1. The term involving y^{10} in the expansion of $(x + y^2)^{12}$ is

$${}_{12}C_5x^7(y^2)^5 \quad \text{or} \quad 792x^7y^{10}.$$

ILLUSTRATION 2. By use of (2),

$$(x + y)^4 = x^4 + {}_4C_1x^3y + {}_4C_2x^2y^2 + {}_4C_3xy^3 + y^4$$
$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

We can prove (1) without reference to Section 104. By definition,

$$(x + y)^n = (x + y)(x + y)\cdots(x + y). \tag{3}$$

The product of the n factors $(x + y)$ on the right in (3) consists of the sum of the results obtained by taking, *in all possible ways*, one term out of each of the factors and multiplying the selected terms. We obtain $x^{n-r}y^r$ in this process by selecting y out of r of the factors $(x + y)$ and x out of the other $(n - r)$ factors. The number of ways of selecting r letters y out of the n factors is ${}_nC_r$ (the number of *combinations* of r letters y out of n letters y, because the *order of selection is of no importance*). Hence, the term $x^{n-r}y^r$ is obtained ${}_nC_r$ times; or, in other words, ${}_nC_r$ is the coefficient of $x^{n-r}y^r$ in the expansion of $(x + y)^n$.

Note 1. If we place $x = 1$ and $y = 1$ in (2), we obtain,

$$2^n = 1 + {}_nC_1 + {}_nC_2 + \cdots + {}_nC_r + \cdots + {}_nC_n,$$

or,
$${}_nC_1 + {}_nC_2 + \cdots + {}_nC_n = 2^n - 1.$$

Thus, *the total number of combinations of n things taken* 1 *at a time, or* 2 *at a time, $\cdots$, or n at a time, is* $(2^n - 1)$.

★EXERCISE 84

Write the expansion of each power by use of the symbol ${}_nC_r$.

1. $(x + y)^7$. **2.** $(x - y)^5$. **3.** $(x^2 - 2y^3)^5$. **4.** $(2x^3 - y^4)^6$.

Find the specified term in the expansion by use of the formula ${}_nC_rx^{n-r}y^r$.

5. $(x - y)^9$; term involving y^5.

6. $(x - 2y^2)^{10}$; term involving y^{12}.

7. $(w + z^{\frac{1}{2}})^{11}$; term involving z^4.

8. $(x^{\frac{1}{2}} + w^2)^8$; term involving $x^{\frac{5}{2}}$.

9. Without computing coefficients, write out the expansion of $(x + y)^{14}$ by use of the symbol ${}_nC_r$. For what value of r is ${}_{14}C_r$ the greatest?

10. For what values of r is ${}_{17}C_r$ greatest?

11. How many committees of one or more people can be made up from a group of 8 people?

12. How many sums of money can be made up by use of combinations of a cent, a nickel, a dime, a quarter, a dollar, a \$5 gold piece, and a \$20 bill?

CHAPTER FIFTEEN

Probability

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187. Mathematical probability. By a **trial** of an event we shall mean an act which gives the event an opportunity to *occur* or to *fail to occur;* when the event *occurs,* we shall call the trial a *success.* In the following definition, we consider an event which, at any trial, can occur in s different ways and fail in f different ways, where *all of these ways are equally likely.*

DEFINITION I. *At any trial of an event, the probability of its occurrence is the ratio of the number of ways of occurrence to the total number of ways of occurrence or failure, or*

$$p = \frac{s}{s+f}; \tag{1}$$

the probability q of the event failing to occur is

$$q = \frac{f}{s+f}. \tag{2}$$

ILLUSTRATION 1. If a bag contains 7 black and 3 white balls, the probability that a ball, drawn at random, will be black is $\frac{7}{10}$, because a black can be drawn in 7 ways ($s = 7$) out of 10 ways of drawing a black or a white.

From (1) and (2) we verify that

$$p + q = 1; \tag{3}$$

or, *the sum of the probabilities of success and of failure is* 1.

In (1) or in (2), the numerator cannot exceed the denominator. Hence, any probability is a positive number not greater than 1. Moreover, if $f = 0$ then $p = 1$; hence, if an event is *certain to occur,* its probability of occurrence is 1. Similarly, if the event is *certain to fail,* its probability of failure is 1 and its probability of success is zero.

Note 1. In this chapter, *to draw,* or *to select,* means to do the act *at random,* without replacements. Also, in a reference to *ways of performing an act,* we shall mean *equally likely ways,* unless otherwise specified.

In solving a problem by use of (1), we first decide on the general *act* whose performance gives the desired event a chance to happen or to fail to happen. Then, we compute separately the number s of *successful cases*, and the *total* number of cases $(s + f)$ for the act, frequently employing the methods of the preceding chapter in this computation.

EXAMPLE 1. A bag contains 5 red and 6 white balls. If we draw 4 balls together, find the probability that two are red and two are white.

SOLUTION. 1. The *general act* is that *we draw 4 balls*. The *total* number of ways $(s + f)$ in which this can be done is ${}_{11}C_4$ or 330.

2. *The successful cases.* We can draw two reds from five in ${}_5C_2$ or 10 ways, and two whites from six in ${}_6C_2$ or 15 ways. Hence, we can draw two red and two white in $10 \cdot 15$ or 150 ways. Therefore, the probability of drawing two red and two white balls is $\frac{150}{330}$ or $\frac{5}{11}$.

188. Meaning of probability as a relative frequency. Consider an event whose probability of occurrence at any trial is p, as defined in Section 187. Suppose that N trials of the event are made, and that S of these are successful. Let us call S/N the *relative frequency* of successes in N trials. Then, a more advanced treatment would show that, if N is *large*, it is very likely that the relative frequency of successes in N trials would be approximately equal to p. Also, for any N trials, it can be proved that the most probable number of occurrences is* pN. We call pN the **expected number** of occurrences in N trials.

ILLUSTRATION 1. Suppose that $\frac{3}{5}$ is the probability that Jones will win a certain game whenever he plays. If, for instance, Jones plays 1000 games, the most probable number of wins is $\frac{3}{5}(1000)$ or 600. Since 1000 is large, we would expect the relative frequency of his wins to be approximately $\frac{3}{5}$. That is, we would expect that the number of games won would differ from 600 by only a small percentage of 600. If he plays only a few games, we would not be surprised if the relative frequency of his wins differed largely from $\frac{3}{5}$. Thus, out of 15 games, the expected number of wins is $\frac{3}{5}(15)$ or 9, but we would not be surprised if he won only 2, or 3, or even none.

189. Experimental or empirical probability. If an event has been observed to occur S times out of N trials, then, until further knowl-

* Or, if pN is not an integer, the most probable number of occurrences is the value (or, in special cases, either of the *two* values) of the integer K which satisfies $pN - q \leqq K \leqq pN + p$.

edge of the event is obtained, we may take S/N as our estimate of the probability that the event will occur at any future trial. Our confidence in this estimate increases as the number N of observed cases increases. We refer to the estimate

$$p = \frac{S}{N}$$

as an experimental probability.

Note 1. Frequently it is impossible or inconvenient to analyze the causes influencing an event so as to determine the possibilities at any trial and how many would be successful. Then, it is impossible to apply the definition of probability given in Section 187, and hence we obtain probabilities experimentally. In such a case, we assume that any event is such that S/N, the relative frequency of successes in N trials, becomes more and more reliable as N increases without bound. Experimental probability is of great importance in the study of statistics.

Note 2. The **American Experience Table of Mortality** (Table III) embodies the results of a large amount of observation of the ages at death of holders of life insurance policies in the United States. The table should be thought of as a record of the year of death for each of 100,000 persons who the table specifies were simultaneously alive at the age of 10 years. The table was arrived at by a process whose discussion is beyond the scope of this text. A mortality table of some sort is essential in the mathematical background of any life insurance company.

EXAMPLE 1. If a man is alive at the age of 25, find the probability that he will live at least 13 years.

SOLUTION. In this, and in similar examples, by a *man* we mean a *man selected at random*. In the American Experience Table of Mortality (Table III) we observe 89,032 men alive at age 25. Of these, 79,611 remain alive at age 38. Hence, the probability of the man aged 25 living at least 13 years is $\frac{79{,}611}{89{,}032}$.

190. Mathematical expectation. If p is the probability of a person receiving a sum $\$S$, we call pS his *mathematical expectation.*

ILLUSTRATION 1. If a man plays a game where the stake is \$100 and where the probability of winning is $\frac{3}{5}$, then his mathematical expectation is $\frac{3}{5}$(\$100) or \$60. Suppose that a professional operator offers to play this game against any man who will pay his expectation as a fee for playing. Then, if many players enter the game, it is probable that about $\frac{3}{5}$ of them will win and hence that the total fees collected will differ from the stakes won from the operator by only a small fraction of the fees. Thus, if 100 players enter, they pay \$6000

in fees; if the expected number $\frac{3}{5}(100)$ or 60 players win, they are paid 60($100) or $6000. To avoid loss, however, the operator should charge *more* than the mathematical expectation as a fee because more than $\frac{3}{5}$ of the players may win. Even with a fee greater than $60, the financial safety of the operator depends on his obtaining a large number of players for his game so that it will be highly probable that the relative frequency of wins S/N will approximate the probability $\frac{3}{5}$ used in computing the lower limit for the fee.

The principle involved in Illustration 1 is of fundamental importance in the conduct of a life insurance company, or of any financial enterprise involving the sharing of natural risks by a group of people. The fact that the risks are unavoidable, instead of being assumed voluntarily as in a game of chance, does not alter the nature of the problem essentially. The financial safety of the operating company demands that the fees charged shall be more than the corresponding mathematical expectations and that a large number of persons should be induced to join the enterprise.

EXERCISE 85

1. A bag contains 4 white, 5 red, and 18 black balls. If one ball is drawn, what is the probability that it will be (*a*) black; (*b*) red or white?

2. The faces of a cubical die are numbered 1, 2, 3, 4, 5, and 6, respectively. If a die is thrown, what is the probability (*a*) that 5 will turn up? (*b*) That 3, or a smaller number, will turn up?

3. As a cooperative class exercise, make 1000 tosses of a coin and keep a record of how many fall heads out of the first (*a*) 10 trials; (*b*) 250 trials; (*c*) 1000 trials. In each case, compare the relative frequency of heads with the probability of a head on a single throw.

4. If the probability of winning a game is $\frac{1}{3}$, what is the probability of losing the game?

5. From a pack of 52 cards, we draw a card. What is the probability that it will be (*a*) a queen? (*b*) An ace or a queen?

Note 1. In connection with Definition 1, page 207, we say that the **odds** are s to f *in favor of the event* if $s > f$, and s to f *against the event* if $s < f$.

6. If the odds are 7 to 3 in favor of a man receiving $50, find (*a*) his probability of receiving $50; (*b*) the probability that he will not receive it; (*c*) his mathematical expectation.

7. If the odds are 4 to 6 against a man receiving $100, find (*a*) his probability of receiving $100; (*b*) the probability that he will not receive it; (*c*) his mathematical expectation.

8. A bag contains 50 envelopes of which 30 contain $5 each, while the others are empty. If I draw an envelope, find my mathematical expectation.

9. A bag contains 5 white and 10 black balls. If 2 balls are drawn together,* find the probability that (a) both are white; (b) both are black; (c) one is white and one is black.

10. A bag contains 4 red and 6 white balls. If 3 balls are drawn together,* find the probability that (a) all are red; (b) one is white and two are red; (c) one is red and two are white.

11. From a group of 6 men and 8 women, a committee of four is chosen by lot. Find the probability that the committee will consist (a) of 4 women; (b) of 2 men and 2 women.

12. A group of 3 letters is chosen at random from the letters a, e, i, o, m, n, and p. Find the probability that (1) the three are vowels; (2) exactly two are vowels.

In Problems 13 *to* 18, *leave each result as a fraction, using the mortality table to find the specified probability.*

13. That a boy aged 12 will be alive 10 years later.

14. That a man aged 33 will live at least 20 years.

15. That a person aged 18 will live at least 10 years.

16. That a person aged 20 will die (a) within 5 years; (b) during the 5th year.

17. That a person aged 22 will die (a) within one year; (b) during his 35th year.

18. That a person aged 62 will die within one year.

19. A bag contains 6 white and 8 black balls. If 3 balls are drawn together, find the probability that (a) all are black; (b) exactly two are white; (c) at least† two are white; (d) all are of the same color.

20. A bag contains 5 white and 10 red balls. If 3 balls are drawn together, find the probability that (a) all are white; (b) exactly one is white; (c) at least† two are white; (d) all are of the same color.

21. There are 5 doors to a certain building. Find the probability that 2 persons, entering at random, will choose the same door.

22. In a single throw with two dice, find the probability of turning up a total (a) of 7; (b) of 10; (c) of at most 4.

23. From a pack of 52 cards, 4 cards are drawn together. Find the probability that all are of the same suit.

* Or, drawn *in succession without replacement.*

† *At least two* are white if (I) *exactly two*, or (II) *exactly three* are white. The *successful* ways, on any trial, are the ways for (I) plus the ways for (II).

24. From a pack of 52 cards, we draw 2 cards together. Find the probability that they consist (a) of an ace and a queen; (b) of an ace and a queen, both of the same suit.

25. The tickets in a box are numbered from 1 to 50 inclusive. If 2 tickets are drawn together, find the probability that the sum of their numbers is even.

26. A first bag contains 6 white and 5 black balls and a second bag contains 4 white and 8 black balls. If 1 ball is drawn from each bag, find the probability that (a) both are white; (b) one is white and one is black.

27. We draw 2 balls together from a bag containing 5 white and 3 red balls, and draw a ball from another bag containing 4 white and 5 black balls. Find the probability that all balls drawn are white.

28. A bag contains 10 black and 4 white balls. If we draw 2 balls in succession, find the probability that both are white, (a) if the first drawn is replaced before the second is drawn; (b) if the first ball is not replaced before the second is drawn.

29. In a single throw with three dice, find the probability of turning up a total (a) of 5; (b) of 6; (c) of at most 6.

30. If 3 balls are drawn in succession from a bag containing 4 white and 6 black balls, find the probability that all are black, if the first drawn is replaced, but the second drawn is not replaced before the next draw.

31. From the integers 1 to 16 inclusive, two different numbers are chosen at random. Find the probability that (a) neither is a multiple of 3; (b) their product is odd; (c) their sum is even.

32. A committee of three is chosen by lot from a group consisting of 4 Americans, 5 Canadians, and 3 Englishmen. Find the probability that (a) all members of the committee are of the same nationality; (b) at least 2 members are English.

33. A committee of four is selected from 4 lawyers, 5 doctors, and 4 professors. Find the probability that (a) at least 2 members of the committee are lawyers; (b) 2 members are from one profession and 2 are from another.

EXAMPLE 1. Six coins are tossed. Find the probability that exactly two will fall heads.

SOLUTION. 1. *The total number of ways* in which 6 coins can fall together is $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$, or 64.

2. *The successful cases.* We can select 2 coins, which are to fall heads, from 6 coins in ${}_6C_2$, or 15, ways. For each of these ways, there is only 1 way in which all of the other 4 coins can fall tails. Hence, there are $15 \cdot 1$, or 15, ways in which 2 heads and 4 tails can result. Therefore, the desired probability is $\frac{15}{64}$.

34. If 5 coins are tossed, find the probability that they will fall (*a*) all heads; (*b*) 3 heads and 2 tails.

35. If 6 coins are tossed, find the probability that they will fall (*a*) all heads; (*b*) 3 heads and 3 tails.

36. If 4 boys and 3 girls take seats at random in a row, find the probability that boys and girls alternate.

37. We form a number of five different digits by use of (1, 2, 3, 4, 5, 6). Find the probability that odd and even digits alternate in the number.

38. In a single throw with 4 coins, find the probability that they will fall with (*a*) exactly 3 heads; (*b*) at least 3 heads.

39. From the mortality table, find the year in which it is most likely that a person now aged 20 will die, and his probability of dying in that year.

40. A three-digit positive integer, without repeated digits, is built at random from the digits 1, 2, 3, 4, 5, 6, and 7. Find the probability that the integer (*a*) is larger than 400; (*b*) is smaller than 260; (*c*) has even and odd digits alternating.

41. If 5 persons take seats at random in a row of 5 chairs, find the probability that a certain two of the persons will be found side by side.

42. If 6 persons take seats at random at a round table, find the probability that a certain two of the persons will be seated side by side.

43. If a hand of 13 cards is drawn from a deck of 52 cards, find the probability that the hand will contain all the kings.

44. A bag contains twenty $1 bills, ten $5 bills, and fifteen $10 bills. What is the mathematical expectation of a person who draws one bill?

191. Probability of mutually exclusive events. Consider a set of mutually exclusive events, any one of which may happen if a certain act is performed.

THEOREM I. *The probability that one or other of a set of mutually exclusive events will occur is the sum of the probabilities of occurrence for the separate events.*

Proof. 1. For simplicity, consider a set of only two events, W and B, and suppose that any trial can result in u different ways, out of which W occurs in m ways and B in n ways.

For concreteness, think of any trial as the drawing of a ball from a bag containing u balls of which m are white and n are black, and consider W and B as the events of drawing a white and a black, respectively.

2. The probability of occurrence of W is $\dfrac{m}{u}$ and of B is $\dfrac{n}{u}\cdot$

3. The m ways in which W occurs and the n ways in which B occurs are all different, because W and B are mutually exclusive. Thus, *one or other* of W and B can happen in $(m + n)$ ways. Hence, the probability of one or other occurring is

$$\frac{m+n}{u}, \quad \text{or} \quad \left(\frac{m}{u}+\frac{n}{u}\right),$$

which is the *sum of the probabilities of occurrence of the separate events.*

ILLUSTRATION 1. A bag contains 3 white, 7 black, and 10 yellow balls. If a ball is drawn, the probability that it is white is $\frac{3}{20}$ and that it is black is $\frac{7}{20}$. Hence, by Theorem I, if a ball is drawn, the probability that it is *either* white *or* black is $\frac{3}{20} + \frac{7}{20}$, or $\frac{10}{20}$, or $\frac{1}{2}$.

192. Dependent events. If the occurrence of one event E_1 affects the probability of another event E_2 happening, then E_2 is said to be *dependent* on E_1.

ILLUSTRATION 1. Consider a bag containing 5 white and 6 black balls. Let E_1 be the event of drawing a *white* ball if one ball is selected at random. Let E_2 be the event of drawing a *black* ball if one ball is selected at random after the first drawing. Then, E_2 depends on E_1. For, if E_1 occurs, the probability of E_2 occurring is $\frac{6}{10}$, while if E_1 fails to occur the probability of E_2 occurring is $\frac{5}{10}$.

THEOREM I. *If the probability of E_1 occurring is p_1 and if, after E_1 has happened, the probability of E_2 occurring is p_2, the probability of E_1 and then E_2 happening is p_1p_2.*

Similarly, *if, after E_2 has happened, the probability of a third event E_3 occurring is p_3, the probability of E_1, and then E_2, and then E_3 happening is $p_1p_2p_3$, etc., for more events.*

Proof. 1. Suppose that any trial of E_1 can result in n_1 ways, in s_1 of which E_1 occurs. **After any trial of E_1,** suppose that any trial of E_2 can result in n_2 ways, and, **if E_1 has already occurred,** that E_2 happens in s_2 of these n_2 ways. At any trial of E_1 and **then** of E_2, the probability of occurrence of E_1 is $p_1 = s_1/n_1$ and of E_2 is $p_2 = s_2/n_2$.

2. By the principle of page 192, E_1 and **then** E_2 can occur in s_1s_2 ways. E_1 can happen or fail and **then** E_2 can happen or fail in n_1n_2 ways. Hence, the probability of E_1 and **then** E_2 occurring is

$$p = \frac{s_1s_2}{n_1n_2} = p_1p_2.$$

ILLUSTRATION 2. Suppose that $\frac{1}{3}$ is the probability of A winning the first game of two to be played, and, if the first is won, that his probability of winning a second is $\frac{2}{5}$; then, his probability of winning both games is $\frac{1}{3}\cdot\frac{2}{5}$, or $\frac{2}{15}$.

EXAMPLE 1. A bag contains 4 white and 7 black balls. If 2 balls are drawn in succession, and not replaced, find the probability that the 1st is black and the 2d is white.

SOLUTION. The probability that the 1st draw will be black is $\frac{7}{11}$. If the 1st draw is black, then there will remain in the bag 4 white and 6 black balls. Hence, the probability that the 2d draw will be white is $\frac{4}{10}$, or $\frac{2}{5}$. By Theorem I, the probability of the 1st draw being black and the 2d white is $\frac{7}{11}\cdot\frac{2}{5}$, or $\frac{14}{55}$.

193. Independent events. We consider a set of events where the happening of any one does not affect the happening of the others.

ILLUSTRATION 1. If E_1 is the event of drawing a white ball from a bag containing n_1 balls of which s_1 are white, and E_2 is the event of drawing a black ball from a second bag containing n_2 balls of which s_2 are black, then E_1 and E_2 are *independent* events. The drawing of a white ball, or the failure to draw one, from the first bag has no effect on E_2.

THEOREM I. *The probability that all of a set of independent events will occur on an occasion when each of them is possible is the product of their separate probabilities of occurrence.*

Note 1. If we consider Theorem I for the case of just two events E_1 and E_2, then we obtain a proof of the theorem by simply *omitting from the proof of Theorem* I, *Section* 192, *all remarks in blackface type.*

ILLUSTRATION 2. If the probability of A winning a certain game is $\frac{3}{10}$, and of B winning another game is $\frac{3}{5}$, then the probability that both A and B will win is $\frac{3}{5}\cdot\frac{3}{10}$, or $\frac{9}{50}$.

EXAMPLE 1. The probability that Jones will win a certain game whenever he plays is $\frac{1}{3}$. If he plays twice, find the probability that he will win one game and lose the other.

SOLUTION. 1. He wins one and loses the other if

(a) *he wins the first game and loses the second;* or if

(b) *he wins the second game and loses the first.*

2. By Theorem I on independent events, the probability of (a) occurring is $\frac{1}{3}\cdot\frac{2}{3} = \frac{2}{9}$, and of (b) occurring is also $\frac{2}{3}\cdot\frac{1}{3} = \frac{2}{9}$. Since (a) and (b) are mutually exclusive, the probability of (a) *or* (b) occurring is $(\frac{2}{9} + \frac{2}{9})$ or $\frac{4}{9}$, by the theorem of Section 191.

Comment. Notice the analysis into *mutually exclusive* subevents.

EXERCISE 86

1. The probability of H winning a game is $\frac{1}{3}$ and of K winning another game is $\frac{1}{5}$. Find the probability that (*a*) both will win; (*b*) H will win and K will lose.

2. In a game where only one man can win, the probability that A will win is $\frac{1}{3}$, and that B will win is $\frac{1}{5}$. Find the probability that one or other of A and B will win.

3. The probability of A arriving in a certain town is $\frac{1}{4}$, and, if he arrives, the probability that he will meet B there is $\frac{1}{3}$. Find the probability that A will arrive and meet B.

4. What is the probability of throwing a tail each time in 4 tosses with a coin?

5. If a coin and a die are tossed, what is the probability of obtaining (*a*) a head, and a 5 on the die; (*b*) a tail, and at least a 3 on the die?

6. In a race, the probability of A winning is $\frac{1}{4}$, and of B winning is $\frac{1}{6}$. Find the probability that A or B will win.

7. Find the probability of throwing a total of 7 each time in 3 tosses with a pair of dice.

8. If a coin is tossed twice, find the probability that one fall is a head and the other a tail.

9. The probability that Jones will win a certain game whenever he plays is $\frac{1}{3}$. If Jones plays twice, find the probability that (*a*) he will win the 1st and lose the 2nd game; (*b*) he will win one game and lose the other.

One bag contains 4 white and 6 black balls, and a second bag contains 3 white and 5 black balls. In Problems 10 to 14 find the probability that the event described will occur.

10. If one ball is drawn from each bag, both balls will be (*a*) white; (*b*) of the same color.

11. If 2 balls are drawn from the first bag, the first being replaced before the second is drawn, then both will be white.

12. If 2 balls are drawn in succession from the second bag, and not replaced, both balls will be black.

13. If a ball is drawn from a bag selected at random, the ball will be white.

HINT. The probability of selecting the 1st bag is $\frac{1}{2}$; after selecting it, the probability of drawing a white from it is $\frac{4}{10}$, or $\frac{2}{5}$. By Section 191, the probability of a white coming from the 1st bag is $\frac{1}{2}\cdot\frac{2}{5}$, or $\frac{1}{5}$.

14. If 2 balls are drawn together from a bag selected at random, both will be black.

15. The probability that a certain man will live 15 years is $\frac{1}{4}$, and that his wife will live 15 years is $\frac{2}{7}$. Find the probability that (*a*) the wife will live, and the man will not live, 15 years; (*b*) one of them will live, and the other will not live, 15 years.

16. K's probability of winning a certain game is $\frac{1}{3}$, and H's probability of winning an independent game is $\frac{1}{2}$. Find the probability that one of them will win and the other will lose.

17. From a deck of 52 cards, we draw 3 cards in succession, replacing each one before the next is drawn. Find the probability that (*a*) all are hearts; (*b*) all are of one suit; (*c*) all are kings or queens.

18. One box contains 5 apples and 10 oranges, and a second box contains 4 apples and 5 oranges. If a man selects a box at random, and then draws one piece of fruit, what is the probability that an apple is drawn?

19. The probability of H winning a certain game, whenever he plays, is $\frac{1}{3}$. If H plays twice, find the probability that he will win at least once.

20. The probability of H winning whenever he plays a certain game is $\frac{1}{4}$. If H plays twice, find the probability that he will lose at most one game.

21. To determine who is to receive a certain prize, A tosses a coin with B, the winner tosses with C, and the winner tosses with D. The winner of the last toss receives the prize. Find each person's probability of receiving the prize.

22. A bag contains 4 white and 6 black balls. A person draws 2 balls, and replaces them by green balls. Then, he draws 2 more balls together. Find the probability that the last two drawn are both of the same color.

★23. Find the probability that, in successive tosses of 2 dice, a total of 5 will be obtained before a 7.

Hint. The favored result is obtained if we get (*a*) 5 on 1st throw; or (*b*) neither 5 nor 7 on the 1st throw and 5 on the 2d throw; or (*c*) neither 5 nor 7 on the first two throws and 5 on the 3d throw; etc. Find the sum of the resulting infinite series.

★24. A man holds 2 kings and a queen out of a deck of 52 cards and then draws 2 more cards from the deck. Find the probability that he will draw 2 queens, or 2 kings, or a king and a queen.

★194. Successive trials of an event.

Example 1. The probability of A winning whenever he plays a certain game is $\frac{1}{4}$. If A plays 6 times, what is his probability of winning four games and losing two?

Solution. 1. Four games, to be won, can be selected from six played in ${}_6C_4$ or 15 mutually exclusive ways.

2. The probability of A losing in a game is $(1 - \frac{1}{4})$, or $\frac{3}{4}$.

3. By Section 193, the probability that A will win any particular four and lose the other two is $\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}$, or $\frac{9}{4096}$; this is the probability of the occurrence of any one of the 15 *mutually exclusive cases* in which A can win four and lose two games. Hence, by Section 191 the probability of *one or other* of the 15 cases occurring is the sum of 15 probabilities (each equal to $\frac{9}{4096}$), or $15 \cdot \frac{9}{4096}$, or $\frac{135}{4096}$.

THEOREM I. *If the probability that an event will occur is p, and that it will not occur is q, at any trial, then the probability that the event will occur just k times out of n trials is ${}_nC_kp^kq^{n-k}$.*

Proof. 1. By Theorem I, Section 193, the probability that any particular k trials will be successful, and the other $(n - k)$ trials will fail, is

$$(p\cdot p\cdots\text{to } k \text{ factors})\cdot[q\cdot q\cdots\text{to } (n-k) \text{ factors}], \quad \text{or} \quad p^kq^{n-k}; \tag{1}$$

in (1), there is one factor p corresponding to each trial we wish to see successful, and a factor q for each of the other trials.

2. It is possible for k successes to occur out of n trials in ${}_nC_k$ different ways. The probability of any one of these mutually exclusive ways occurring is p^kq^{n-k}; hence, by Section 191, the probability that exactly k successes will occur in one or other of these ${}_nC_k$ ways is ${}_nC_kp^kq^{n-k}$.

COROLLARY I. *The probability that the event will happen at least k times out of n trials is*

$$p^n + {}_nC_{n-1}p^{n-1}q + {}_nC_{n-2}p^{n-2}q^2 + \cdots + {}_nC_kp^kq^{n-k}. \tag{2}$$

Proof. In (2), the 1st term is the probability of exactly n successes in n trials, the next term is the probability of just $(n - 1)$ successes, etc., the last term is the probability of just k successes in n trials. By Section 191, the sum in (2) is the probability of *k or more* successes in n trials.

Note 1. From (1) page 205, recognize that ${}_nC_kp^kq^{n-k}$ is the term in the expansion of $(p + q)^n$ which contains p^k as a factor. Also, (2) consists of *the first $(n - k + 1)$ terms in the expansion of $(p + q)^n$.*

Note 2. The most probable number of successes in n trials is the value of k for which ${}_nC_kp^kq^{n-k}$ is greatest. An investigation of the character of this expression leads to the fact, stated in Section 188, that *the most probable number of successes is np (approximately).*

EXERCISE 87

1. The probability of H winning whenever he plays a certain game is $\frac{1}{5}$. If H plays 5 times, find the probability that he will win (*a*) exactly twice; (*b*) at least twice; (*c*) at most twice.

2. The probability of K winning whenever he plays a certain game is .4. If K plays 6 times, find the probability that he will win (*a*) exactly 4 games; (*b*) at least 4 games; (*c*) at most 3 games.

3. Find the probability that, in 6 tosses with a coin (or, in one toss with 6 coins), (*a*) exactly two will fall heads; (*b*) at least two will fall heads; (*c*) at most two will fall heads.

4. If 3 dice are tossed, what is the probability that (*a*) exactly two will give aces; (*b*) at least two will give aces?

5. A bag contains 4 white and 2 black balls. If we draw 5 balls in succession, replacing each one before the next is drawn, what is the probability that, of those drawn, (*a*) exactly three are white; (*b*) at least three are white?

6. What is the probability of throwing at least three sevens in five throws with a pair of dice?

7. From a group of 3 girls and 2 boys, we make 5 random selections of a committee of two persons. What is the probability that exactly 3 of the five committees will consist entirely of boys?

8. A bag contains 5 black and 4 white balls. We draw 2 balls together from the bag, then replace these balls, and repeat the process until 4 drawings of 2 balls each have been made. What is the probability that exactly 3 drawings give 1 black and 1 white ball?

9. A bag contains 5 red balls, 3 green balls, and 7 white balls. We draw a ball, 5 times in succession, replacing each ball before the next drawing. What is the probability that we draw 3 green and 2 red balls?

10. Find the probability that, of four persons, each of age 20, exactly three will remain alive 15 years from now.*

11. Find the probability that, out of three classmates, each of age 24, at least two will be alive 30 years from now.*

12. Find the probability of obtaining the most probable number of heads, if a coin is tossed 7 times.

13. Find the probability of obtaining the most probable number of aces, if a die is tossed five times.

14. If p is the probability of success on a single trial, show that the probability of at least one failure in m trials is $(1 - p^m)$.

15. State and prove a theorem like that of Problem 16, about at least one success in m trials.

* Use the mortality table. Compute by logarithms or give factored form.

CHAPTER SIXTEEN

Mathematical Induction

195. Mathematical induction is a method of proof which we shall illustrate in establishing the theorems stated in the following examples.

EXAMPLE 1. *If n is any positive integer, prove that*

$$2 + 4 + 6 + \cdots + 2n = n(n + 1), \tag{1}$$

or, the sum of the first n positive even integers is $n(n + 1)$.

Note 1. There are n terms on the left in equation 1, and $2n$ is a formula by means of which any term can be computed. Thus, if n is 4 or greater in (1), the 4th term is 2(4) or 8.

Proof. I. *Verification of first few special cases.* By substitution in equation 1, we obtain the following results.

When we place $n = 1$, $\qquad 2 = 1(1 + 1),$ or $2 = 2.$

When we place $n = 2$, $\qquad 2 + 4 = 2(2 + 1),$ or $6 = 6.$

When we place $n = 3$, $\qquad 2 + 4 + 6 = 3(3 + 1),$ or $12 = 12.$

Hence, equation 1 is true when n is 1, 2, or 3. (The verification of these special cases may create a *presumption* that all cases of the theorem are true but, *by itself, the verification of any number of special cases does not prove a general theorem.*)

II. AUXILIARY THEOREM. *If k is any value of n for which equation 1 is true, then the equation is true also when $n = k + 1$.*

Proof. 1. By hypothesis, equation 1 is true when $n = k$, or

$$2 + 4 + 6 + \cdots + 2k = k(k + 1). \tag{2}$$

By use of (2) we wish to prove that

$$\begin{aligned} 2 + 4 + 6 + \cdots + 2(k + 1) &\overset{?}{=} (k + 1)[(k + 1) + 1] \\ &\overset{?}{=} (k + 1)(k + 2), \end{aligned} \tag{3}$$

which results from (1) when $n = k + 1$. In (3), we place "?" over "=" because the equality is not yet proved. Recognize that

$$2 + 4 + 6 + \cdots + 2(k + 1) = 2 + 4 + 6 + \cdots + 2k + 2(k + 1), \tag{4}$$

since the sum of $(k+1)$ terms equals the sum of k terms plus the $(k+1)$th term. By use of (4), the equation 3 which we wish to establish becomes

$$2+4+6+\cdots+2k+2(k+1) \stackrel{?}{=} (k+1)(k+2). \tag{5}$$

Our *hypothesis* is (2); by means of it we desire *to prove* (5).

2. Add $2(k+1)$ to both sides of equation 2:

$$2+4+6+\cdots+2k+2(k+1) = k(k+1)+2(k+1)$$

(factoring)

$$= (k+1)(k+2). \tag{6}$$

Hence, each side in (6) is the same as the corresponding side of (5). Therefore, we have shown that (5) is true if (2) is true, which completes the proof of the Auxiliary Theorem.

Conclusion. Let H be any positive integer. Then, to show that equation 1 is true when $n = H$, we first recall that (1) is true, by verification, when n equals 1, or 2, or 3. Now, by use of Part II, since equation 1 is true when $n = 3$, it follows that equation 1 is true when $n = 3 + 1 = 4$. By use of Part II again, since equation 1 is now known to be true when $n = 4$, hence equation 1 is true also when $n = 4 + 1 = 5$. On proceeding step by step in this manner, by successive applications of Part II, we eventually reach the conclusion that equation 1 is true when $n = H$. Since H represented any positive integer, we have proved Theorem I.

Comment. It is essential to recognize that (5) was not used in the proof leading to (6). We wrote (5) merely to *guide* our proof.

EXAMPLE 2. *If n is any positive integer, prove that*

$$1\cdot 2+2\cdot 3+\cdots+n(n+1) = \tfrac{1}{3}n(n+1)(n+2). \tag{7}$$

Proof. I. Equation 1 is true when $n = 1$, 2, or 3 because

when $n = 1$, $\quad 1\cdot 2 = \frac{1}{3}(1)(1+1)(1+2)$, or $2 = 2$;

when $n = 2$, $\quad 1\cdot 2+2\cdot 3 = \frac{1}{3}(2)(2+1)(2+2)$, or $8 = 8$;

when $n = 3$, $\quad 1\cdot 2+2\cdot 3+3\cdot 4 = \frac{1}{3}(3)(3+1)(3+2)$, or $20 = 20$.

II. AUXILIARY THEOREM. *If k is any value of n for which equation 7 is true, then the equation is true also when $n = k+1$.*

Proof. 1. By hypothesis, (7) is true when $n = k$, or

$$1\cdot 2+2\cdot 3+\cdots+k(k+1) = \tfrac{1}{3}k(k+1)(k+2). \tag{8}$$

Under this hypothesis we wish to show the truth of the following equation, which is obtained by placing $n = k+1$ in (7):

$$\begin{aligned} &1\cdot 2+2\cdot 3+\cdots+(k+1)[(k+1)+1] \\ &\qquad \stackrel{?}{=} \tfrac{1}{3}(k+1)[(k+1)+1][(k+1)+2]. \end{aligned} \tag{9}$$

2. On explicitly indicating the kth term on the left in (9), we may rewrite (9) as follows:

$$1 \cdot 2+2 \cdot 3+\cdots+k(k+1)+(k+1)(k+2) \stackrel{?}{=} \tfrac{1}{3}(k+1)(k+2)(k+3). \quad (10)$$

To establish (10), add $(k+1)(k+2)$ to both sides of (8); we obtain

$$\begin{aligned} 1 \cdot 2 + 2 \cdot 3 + \cdots + k(k+1) + (k+1)(k+2) \\ = \tfrac{1}{3}k(k+1)(k+2) + (k+1)(k+2). \end{aligned} \quad (11)$$

When the right member of (11) is simplified, it becomes

$$\frac{k(k+1)(k+2)+3(k+1)(k+2)}{3} = \frac{1}{3}(k+1)(k+2)(k+3).$$

Since each side of (11) is the same as the corresponding side of (10), hence we have shown that (9) is true if (8) is true.

Conclusion. The student should now supply a concluding statement.

In order that mathematical induction may be used, it must be possible to arrange the cases with which the theorem deals in a definite order, so that there is a 1st case, a 2d case, $\cdots$, a kth case, $\cdots$. The total number of cases may be large or small, but we are interested in the method mainly for use in case a theorem deals with infinitely many cases.

196. Outline of a proof by mathematical induction.

PART I. *Verification of the truth of the theorem in the first few special cases.*

PART II. Proof of an auxiliary theorem. *If the* kth *case of the theorem is true, then the* $(k+1)$th *case is also true.*

CONCLUSION. *The argument for concluding that all cases are true.*

Note 1. An argument by mathematical induction may be compared to climbing a ladder, where each round corresponds to a special case of the theorem. In the proof, Part I shows that we are able to climb onto the first round of the ladder. Part II proves that we are able to pass from round to round and, hence, that we are able to arrive eventually at any round, however high.

Note 2. In Part I of a proof by mathematical induction, it usually suffices logically to verify only the first case of the theorem. However, the meaning of the theorem will always be more thoroughly appreciated if several cases are verified.

Note 3. Both Part I and Part II of a proof by mathematical induction are necessary. Thus, Part I by itself would be insufficient because, even though many special cases of a theorem are true, nevertheless the theorem may not be true in all cases. Thus, it can be verified that $(n^2 - n + 41)$

is a prime number for each of the values $n = 1, 2, 3, \cdots, 40$. Hence, it might be inferred that $(n^2 - n + 41)$ is a prime number for *all* values of the integer n. However, this general result is not true because, on substituting $n = 41$, we find that $(n^2 - n + 41)$ becomes $[(41)^2 - 41 + 41]$, or $(41)^2$, and $(41)^2$ is *not* a prime number.

Likewise, Part II must be accompanied by Part I in order to furnish a proof. Thus, if we should forget the necessity for the verification of special cases in Part I, we could apparently prove the *false* statement that, *if n is any positive integer, then*

$$2 + 4 + 6 + \cdots + 2n = 20 + n(n + 1). \tag{1}$$

When we compare this equation with the correct equation of Section 195, we see that equation 1 is not true for *any* value of n. Nevertheless, we can easily prove the auxiliary theorem of Part II, which would state that, if

$$2 + 4 + 6 + \cdots + 2k = 20 + k(k + 1), \tag{2}$$

then $$2 + 4 + 6 + \cdots + 2(k + 1) = 20 + (k + 1)(k + 2).$$

This result can be proved by adding $2(k + 1)$ to both sides of (2).

Note 4. In the natural sciences a general conclusion is often reached, although not demonstrated in the mathematical sense, by a consideration of what happens in a number of special cases. Such reasoning is called *ordinary*, or *incomplete*, induction. In contrast to it, mathematical induction is often called *complete* induction.

The student must not infer that mathematical induction applies only when the theorem is stated by means of an *equation*. In the following Example 1, no equation is involved in the theorem.

EXAMPLE 1. *Prove that, if n is any positive integer, then $x^{2n} - y^{2n}$ has $x + y$ as a factor.*

SOLUTION. I. When $n = 1$, $x^{2n} - y^{2n}$ becomes $x^2 - y^2$, which is seen to have $x + y$ as a factor. If $n = 2$, $x^{2n} - y^{2n}$ becomes $x^4 - y^4$, which has $x + y$ as a factor:

$$x^4 - y^4 = (x + y)(x^3 - x^2y + xy^2 - y^3).$$

II. AUXILIARY THEOREM. *If $x^{2n} - y^{2n}$ has $x + y$ as a factor when $n = k$, then $x^{2n} - y^{2n}$ has $x + y$ as a factor also when $n = k + 1$.*

Proof. 1. If $x^{2n} - y^{2n}$ has $x + y$ as a factor when $n = k$, then

$$x^{2k} - y^{2k} = (x + y)F, \tag{3}$$

where we let F represent the other factor. When $n = k + 1$, $x^{2n} - y^{2n}$ becomes $x^{2k+2} - y^{2k+2}$. We verify that *

$$x^{2k+2} - y^{2k+2} = x^2(x^{2k} - y^{2k}) + y^{2k}(x^2 - y^2). \tag{4}$$

* The identity 4 is obtained on dividing $x^{2k+2} - y^{2k+2}$ by $x^{2k} - y^{2k}$; the quotient is x^2 and the remainder is $y^{2k}x^2 - y^{2k+2}$, or $y^{2k}(x^2 - y^2)$.

2. Hence, by use of equations 3 and 4, we obtain

$$x^{2k+2} - y^{2k+2} = x^2(x + y)F + y^{2k}(x + y)(x - y)$$
$$= (x + y)[x^2F + y^{2k}(x - y)].$$

Therefore, $x^{2k+2} - y^{2k+2}$ has the factor $x + y$ if $x^{2k} - y^{2k}$ has the factor $x + y$, and hence the auxiliary theorem has been proved. The student should supply the concluding statement for the solution.

EXERCISE 88

By use of mathematical induction, prove that the equation, or statement, is true for all positive integral values of n. Do not use formulas previously proved.

1. $4 + 8 + 12 + \cdots + 4n = 2n(n + 1)$.

2. $1 + 2 + 3 + \cdots + n = \dfrac{n(n + 1)}{2}$.

3. The sum of the first n positive integral multiples of 3 is $\frac{3}{2}n(n + 1)$.

4. The sum of the first n positive integral multiples of 6 is $3n(n + 1)$.

5. $1 + 3 + 5 + \cdots + (2n - 1) = n^2$. (State this theorem in words.)

6. $1 + 4 + 7 + \cdots + (3n - 2) = \frac{1}{2}n(3n - 1)$.

7. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n}{6}(n + 1)(2n + 1)$.

8. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2}{4}(n + 1)^2$.

9. $2\cdot 4 + 4\cdot 6 + 6\cdot 8 + \cdots + 2n(2n + 2) = \dfrac{n}{3}(2n + 2)(2n + 4)$.

10. $1 + 3 + 6 + \cdots + \dfrac{n}{2}(n + 1) = \dfrac{n}{6}(n + 1)(n + 2)$.

11. $\dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \dfrac{1}{3\cdot 4} + \cdots + \dfrac{1}{n(n + 1)} = \dfrac{n}{n + 1}$.

12. $2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2$.

13. $3 + 3^2 + 3^3 + \cdots + 3^n = \frac{1}{2}(3^{n+1} - 3)$.

14. $1 + 5 + 5^2 + \cdots + 5^{n-1} = \frac{1}{4}(5^n - 1)$.

15. The sum of the cubes of the first n positive even integers is $2n^2(n + 1)^2$.

16. $a + ar + ar^2 + \cdots + ar^{n-1} = \dfrac{a - ar^n}{1 - r}$.

17. $a + (a + d) + (a + 2d) + \cdots + [a + (n - 1)d] = \dfrac{n}{2}[2a + (n - 1)d]$.

18. If n is a positive integer, then $(x^n - y^n)$ has $(x - y)$ as a factor. [Notice that $(x^{r+1} - y^{r+1}) = x(x^r - y^r) + y^r(x - y)$.]

19. If n is a positive integer, $(x^{2n-1} + y^{2n-1})$ has $(x + y)$ as a factor.

★197. Binomial Theorem. *The binomial formula,*

$$\left.\begin{aligned}(x+y)^n &= x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \cdots\\ &+ \frac{n(n-1)\cdots(n-r+2)}{(r-1)!}x^{n-r+1}y^{r-1}\\ &+ \frac{n(n-1)\cdots(n-r+1)}{r!}x^{n-r}y^r + \cdots + y^n,\end{aligned}\right\} \tag{1}$$

for the expansion of $(x+y)^n$ is true for every positive integral value of n.

Note 1. In (1), we repeated formula 3 of page 101, and, for later convenience, inserted the rth term, involving $x^{n-r+1}y^{r-1}$, as well as the $(r+1)$th term, involving $x^{n-r}y^r$.

Proof. I. In previous problems, we have verified (1) for $n = 1, 2, 3$, and 4 as well as for numerous other special cases.

II. Auxiliary Theorem. *If the binomial formula is true when $n = k$, then the formula is true also when $n = k + 1$.*

Proof. 1. By hypothesis, (1) is true if $n = k$, or

$$(x+y)^k = x^k + kx^{k-1}y + \frac{k(k-1)}{2!}x^{k-2}y^2 + \cdots \tag{2}$$

$$+ \frac{k(k-1)\cdots(k-r+2)}{(r-1)!}x^{k-r+1}y^{r-1} + \frac{k(k-1)\cdots(k-r+1)}{r!}x^{k-r}y^r + \cdots + y^k.$$

Under this assumption, we wish to prove that $(x+y)^{k+1}$ is equal to the following expression, which we write on substituting $n = k + 1$ in the right member of (1):

$$\left.\begin{aligned}&x^{k+1} + (k+1)x^ky + \frac{(k+1)k}{2!}x^{k-1}y^2 + \cdots\\ &\quad + \frac{(k+1)(k)\cdots[(k+1)-r+1]}{r!}x^{(k+1)-r}y^r + \cdots + y^{k+1}.\end{aligned}\right\} \tag{3}$$

2. Since $(x+y)^{k+1} = (x+y)(x+y)^k$, we can obtain $(x+y)^{k+1}$ by multiplying the right member of (2) by $(x+y)$. On multiplying, the first few terms obtained are as follows:

$$\left.\begin{aligned}&x^{k+1} + kx^ky + \frac{k(k-1)}{2!}x^{k-1}y^2 + \cdots\\ &\quad + x^ky + kx^{k-1}y^2 + \cdots.\end{aligned}\right\} \tag{4}$$

In (4), the first line results from use of the x of the multiplier $(x+y)$ and the second line from use of the y. On adding similar terms in these two lines, we obtain

$$x^{k+1} + (k+1)x^k y + \frac{k(k+1)}{2!} x^{k-1}y^2 + \cdots,$$

where we find the same first three terms as in (3).

3. We must also show that, in the result of the multiplication by $(x + y)$, the general term involving y^r is identical with the corresponding term in (3). In multiplying $(x + y)^k$ by $(x + y)$, we obtain terms involving y^r by multiplying

(*a*) the term in y^r in (2) by the x of $(x + y)$, and

(*b*) the term in y^{r-1} in (2) by the y of $(x + y)$.

That is, we obtain, as the terms in y^r,

$$\frac{k(k-1)\cdots(k-r+1)}{1\cdot 2\cdots r} x^{k-r+1}y^r + \frac{k(k-1)\cdots(k-r+2)}{1\cdot 2\cdots(r-1)} x^{k-r+1}y^r.$$

On reducing these to a common denominator and adding, we obtain

$$\frac{[(k-r+1)+r](k)(k-1)\cdots(k-r+2)}{1\cdot 2\cdots r} x^{k-r+1}y^r. \tag{5}$$

Since $k - r + 2 = [(k+1) - r + 1]$, hence (5) can be written

$$\frac{(k+1)(k)(k-1)\cdots[(k+1)-r+1]}{r!} x^{(k+1)-r}y^r,$$

which is the same as the corresponding term in (3).

4. On multiplying y^k in (2) by the y of $(x + y)$, we obtain y^{k+1}, which is the last term in (3). Hence, we have shown that, if (2) is true, then $(x+y)^{k+1}$ is given by (3). Hence, the auxiliary theorem is proved. The reader should make the concluding statement for the proof.

Note 2. The conclusion of a proof of a theorem by mathematical induction can be written in various equally logical ways. Let the general case of the theorem be called the nth case (n a positive integer), suppose that in Part I the theorem has been verified at least for $n = 1$, and assume that Part II of the proof has been completed. Then, instead of writing the conclusion as on page 221, we may use the following argument, which employs the method of *indirect proof*, consisting of an assumption that the theorem is *false* and then the arrival at a *contradiction*.

Conclusion. To prove that the theorem holds for *all* values of n, *assume that this is not true.* Then, let $(k + 1)$ represent the *smallest* value of n for which the theorem is *false*. By Part I of the proof, $(k + 1)$ is at least 2, and the theorem *fails* when $n = k + 1$ but *holds* when $n = k$, because $(k + 1)$ was the *smallest* value of n for which the theorem is *false*. The preceding sentence *contradicts* the result of Part II of the proof. Thus, the original *assumption* above that the theorem is *not true for all values of n* is *false*. Hence, the theorem *is true for all values of n.*

CHAPTER SEVENTEEN

Determinants

198. Inversions. In considering any permutation of positive integers, we shall say that there is an *inversion* whenever a number precedes (is to the left of) one which is smaller.

ILLUSTRATION 1. In (1, 5, 3, 2, 4) there are four inversions because 5 precedes 3, 2, and 4, and 3 precedes 2.

199. Determinants of any order. We shall give a definition of a determinant of *any* order which will include as special cases the definitions of determinants of the 2d and 3d orders on pages 60 and 61.

Consider an array of n^2 numbers, called **elements,** arranged in n rows and n columns of n numbers each, and inclosed by two vertical bars. Represent each element by a letter with a subscript showing the number of the row where the element lies, and use the same letter for all elements in the same column.

ILLUSTRATION 1. We indicate an array of 4^2 or 16 elements by the symbol at the left and an array of n^2 elements by the symbol at the right below, where the n column letters would be chosen without duplications:

$$\text{I.}\quad \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}. \qquad\qquad \text{II.}\quad \begin{vmatrix} a_1 & b_1 & \cdots & r_1 \\ a_2 & b_2 & \cdots & r_2 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_n & b_n & \cdots & r_n \end{vmatrix}.$$

DEFINITION I. *A square array of n^2 numbers, such as* (II), *is called a* **determinant of the nth order.** *It is a symbol for the sum of all terms which can be formed*

1. *by taking as factors one and only one element from each row, and from each column, and*

2. *by giving to each such product* **a plus or a minus sign** *according as the number of inversions in the subscripts is* **even or odd,** *after the letters of the product are written* **in the order in which they appear in the first row of the determinant.**

ILLUSTRATION 2. We obtain the products in the expansion of the following determinant by attaching the subscripts 1, 2, and 3, in every possible order, to the type product "$a\,b\,c$." We thus obtain $a_1b_2c_3$, $a_3b_2c_1$, etc. We determine the sign to be attached to any product by Step 2 of Definition I. Thus, we give $a_2b_3c_1$ a *plus* sign because (2, 3, 1) shows *two* inversions, and $a_2b_1c_3$ a *minus* sign because (2, 1, 3) shows *one* inversion.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3.$$

This expression is the same as that used on page 61 as a definition of the determinant of the 3d order. Hence, Definition I includes the previous definition as a special case. More easily, the student can verify that Definition I includes as a special case our previous definition of a determinant of the 2d order.

The sum of the signed products described in Definition I is called the **expansion** of the determinant.

The **principal diagonal** of a determinant is the line of elements from the upper left- to the lower right-hand corner.

THEOREM I. *The expansion of a determinant of order n contains $n\,!$ terms.*

Proof. The number of terms is the same as the number of ways in which we can attach the subscripts $1, 2, \cdots, n$ to the n letters used to denote columns. This number of ways is ${}_nP_n$, or $n\,!$.

ILLUSTRATION 3. The expansion of a determinant of the 4th order contains 4!, or 24 terms.

Note 1. In a permutation of letters $a, b, c, \cdots$, we shall say there is an inversion *whenever one letter precedes another which follows it in alphabetical order.* Suppose, now, that in a determinant we use the same letter for all elements of the same *row*, with the row-letters in alphabetical order from top to bottom, and subscripts to distinguish the columns. Then, we shall replace Step 2 of Definition I by the following statement: *give to each product a plus or a minus sign according as the number of inversions among the letters is even or odd, after the factors of the product are written so that the* **subscripts appear in the natural order** (1, 2, 3, $\cdots$).

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

EXERCISE 89

Determine the number of inversions in each arrangement:

1. 5, 3, 4, 1, 7, 9. **2.** 6, 1, 4, 3, 2, 5. **3.** 1, 4, 3, 2, 5. **4.** 3, 4, 1, 2, 5.

5. By use of Definition I, write out the expansion of the determinant of order 4 in Illustration 1, Section 199.

200. Properties of determinants.

PROPERTY I. *The value of a determinant is not changed if corresponding rows and columns are interchanged.**

ILLUSTRATION 1. If we let

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad \text{and} \quad D' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix},$$

then Property I states that $D = D'$. The reader can verify this special case of Property I by expanding D and D', by the method of page 61.

From Property I, it follows that, *for every theorem concerning the columns of a determinant, there is a corresponding theorem concerning the rows.* Hence, we shall state the following properties as true for both rows and columns, but shall prove only the parts referring to columns.

PROPERTY II. *If all elements of a column (or row) of a determinant D are multiplied by the same number k, the value of the determinant is multiplied by k.*

Proof. One and only one element of the column is a factor of each term of D. Hence, if each element of the column is multiplied by k, we obtain a new determinant D' each of whose terms is k times a corresponding term of D. Therefore $D' = kD$.

ILLUSTRATION 2. By Property II, $$k\begin{vmatrix} 3 & 4 \\ 5 & 7 \end{vmatrix} = \begin{vmatrix} 3 & 4k \\ 5 & 7k \end{vmatrix}.$$

From Illustration 2, it is evident that Property II justifies

PROPERTY III. *A common factor of all elements of a column (or row) may be removed and written before the determinant.*

PROPERTY IV. *If all elements of a column (or row) are zero, the value of the determinant is zero.*

Proof. Each term of the expansion is zero, because each term contains one factor from the column of zeros.

PROPERTY V. *If two columns (or rows) are interchanged, the sign of the determinant is changed.**

* For a proof, see Appendix, Note 6.

ILLUSTRATION 3. If $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D' = \begin{vmatrix} c_1 & b_1 & a_1 \\ c_2 & b_2 & a_2 \\ c_3 & b_3 & a_3 \end{vmatrix}$,

then Property V states that $D = -D'$. The reader can verify this equality by expanding D and D'; each term of D' is the negative of a term of D.

PROPERTY VI. *If two columns (or rows) of a determinant D are identical, then $D = 0$.*

ILLUSTRATION 4. By Property VI, $\begin{vmatrix} a & x & x \\ b & y & y \\ c & z & z \end{vmatrix} = 0.$

Proof. Interchange the identical columns in D. By Property V, the value of the new determinant is $-D$. But, since the columns were identical, the new determinant is the same as D. Hence, $D = -D$, or $2D = 0$, or $D = 0$.

PROPERTY VII. *If each element of some column (or row) is expressed as the sum of two, or more, numbers, the determinant may be expressed as the sum of two, or more, determinants.*

ILLUSTRATION 5. $\begin{vmatrix} a_1 & (b_1 + d_1) & c_1 \\ a_2 & (b_2 + d_2) & c_2 \\ a_3 & (b_3 + d_3) & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}.$

This special case is seen to be true because each term of the determinant on the left is the sum of the corresponding terms of the determinants on the right. For instance, $a_1(b_3 + d_3)c_2 = a_1b_3c_2 + a_1d_3c_2$. A similar relation establishes Property VII for a determinant of any order.

PROPERTY VIII. *The value of a determinant is not changed if to each element of any column (or row) we add k times the corresponding element of some other column (or row).*

Proof. 1. For convenience in details, consider only the special case which states that D and D' below are equal.

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \qquad D' = \begin{vmatrix} a_1 & (b_1 + kc_1) & c_1 \\ a_2 & (b_2 + kc_2) & c_2 \\ a_3 & (b_3 + kc_3) & c_3 \end{vmatrix}.$$

2. We apply Property VII and then Property III to D':

$$D' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & kc_1 & c_1 \\ a_2 & kc_2 & c_2 \\ a_3 & kc_3 & c_3 \end{vmatrix} = D + k\begin{vmatrix} a_1 & c_1 & c_1 \\ a_2 & c_2 & c_2 \\ a_3 & c_3 & c_3 \end{vmatrix}.$$

3. By Property VI, the last determinant is zero; hence, $D' = D$.

DEFINITION I. *In any determinant, if the row and column containing a given element, say* ϵ, *are blotted out, the determinant formed from the remaining elements is called the* **minor** *of* ϵ.

ILLUSTRATION 6. In $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, the minor of c_2 is $\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$.

PROPERTY IX. *In any determinant* D, *if* a_1 *is the element in the upper left-hand corner and if* A_1 *is its minor, then the terms of* D *involving* a_1 *are given by* a_1A_1.*

ILLUSTRATION 7. Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$; then, $A_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, or, $A_1 = b_2c_3 - b_3c_2$. In the expansion of D, on page 228, the terms involving a_1 are $(a_1b_2c_3 - a_1b_3c_2)$, or $a_1(b_2c_3 - b_3c_2)$, which is a_1A_1.

PROPERTY X. **Terms involving any element.** *In any determinant* D, *if* ϵ *is the element in the* hth *row and* kth *column, and if* E *is the minor of* ϵ, *the terms of* D *involving* ϵ *are* $+\epsilon E$ *or* $-\epsilon E$ *according as* $(h + k)$ *is even or odd.*

ILLUSTRATION 8. In D of Illustration 7, if ϵ is the element c_2, then $h = 2$ and $k = 3$; $h + k = 5$, which is odd. By (X), the terms of D involving c_2 are

$$-c_2\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} = -c_2(a_1b_3 - a_3b_1). \tag{1}$$

Proof. 1. Interchange the hth row with the one above it, then with the new one above it, etc., until, after $(h - 1)$ interchanges, the original hth row becomes the 1st row. Then interchange the kth column with the one at its left, then with the new column at its left, etc., until, after $(k - 1)$ interchanges, the original kth column becomes the 1st column. Call the final determinant D'; ϵ *is in the upper left-hand corner of* D'.

2. In D', the minor of ϵ is E, because on blotting out the elements of the 1st row and the 1st column of D' we leave the elements of E in the order they had originally in D. Hence, by Property IX, the terms of D' involving ϵ are given by ϵE.

* For a proof of Property IX, see Appendix, Note 6.

3. D' was obtained by starting with D and performing the $[(h-1)+(k-1)]$, or $(h+k-2)$, interchanges of Step 1. By Property V, each interchange changes the signs of all terms. Hence, since ϵE gives the terms of D' involving ϵ, then the terms of D which involve ϵ are $+\epsilon E$ or $-\epsilon E$ according as $(h+k-2)$ is even or odd, or, according as $(h+k)$ is even or odd.

201. Method for expanding a determinant D by minors, according to the elements of a given column (or row).

1. *Multiply each element of the column by its minor, and give the product a* **plus** *or a* **minus** *sign according as the sum of the numbers of the row and the column containing the element is* **even** *or* **odd.**

2. *Take the sum of these signed products. This sum is D.*

Proof. Each term in the expansion of D contains one and only one element of the given column. Hence, the expansion is the sum of the terms involving the 1st element of the column, plus the terms involving the 2d element, etc., and, by Property X, these are the terms described in Step 1 of the method.

ILLUSTRATION 1. In the following equality, we expand the determinant of the 3d order according to the elements of the 1st column:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}.$$

This equality can be verified by expanding all determinants above by the methods of pages 60 and 61.

ILLUSTRATION 2. Expanding according to the elements of the 2d row,

$$\begin{vmatrix} 3 & -1 & 5 & 0 \\ -1 & 0 & -4 & 2 \\ 2 & 3 & -2 & 6 \\ 4 & -2 & -3 & -1 \end{vmatrix} =$$

$$-(-1)\begin{vmatrix} -1 & 5 & 0 \\ 3 & -2 & 6 \\ -2 & -3 & -1 \end{vmatrix} + 0 - (-4)\begin{vmatrix} 3 & -1 & 0 \\ 2 & 3 & 6 \\ 4 & -2 & -1 \end{vmatrix} + 2\begin{vmatrix} 3 & -1 & 5 \\ 2 & 3 & -2 \\ 4 & -2 & -3 \end{vmatrix}.$$

202. Cofactors. In a determinant, if ϵ is the element which stands in row h and column k, and if E is the minor of ϵ, the *cofactor* of ϵ is defined as $+E$ or $-E$ according as the sum $(h+k)$ of the row and column numbers of ϵ is *even* or *odd.*

ILLUSTRATION 1. If ϵ stands in row 3 and column 6, the cofactor of ϵ is *minus* the minor of ϵ, because $(3 + 6)$ is odd.

If the notion of a cofactor is employed, the rule for the expansion of a determinant can be restated as follows.

The value of a determinant is the sum obtained by multiplying each element of any row (or column) by its cofactor and adding the results.

203. To evaluate a determinant of the second order, use the method of Section 68. If the order of a determinant is greater than 2, it is usually most convenient to proceed as follows:

1. *By use of Property* VIII, *reduce all, except at most two, of the elements of a certain column (or row) to zeros.*

2. *Expand the resulting determinant by minors, according to the elements of the column (or row) containing the zeros. For each minor, if its order is greater than* 2, *proceed as in Step* 1.

EXAMPLE 1. Compute the value of
$$D = \begin{vmatrix} 5 & 7 & 8 & 6 \\ 11 & 16 & 13 & 11 \\ 14 & 24 & 20 & 23 \\ 7 & 13 & 12 & 2 \end{vmatrix}.$$

SOLUTION. 1. Subtract the 1st row from the last, twice the 1st row from the 2d row, and three times the 1st row from the 3d row. Then,
$$D = \begin{vmatrix} 5 & 7 & 8 & 6 \\ 1 & 2 & -3 & -1 \\ -1 & 3 & -4 & 5 \\ 2 & 6 & 4 & -4 \end{vmatrix} = 2\begin{vmatrix} 5 & 7 & 8 & 6 \\ 1 & 2 & -3 & -1 \\ -1 & 3 & -4 & 5 \\ 1 & 3 & 2 & -2 \end{vmatrix};$$
in obtaining the last determinant, Property III was used in removing the factor 2 from the elements of the last row of the preceding determinant.

2. In the last determinant, subtract the 2d row from the last row, and five times the 2d row from the 1st row; add the 2d row to the 3d row; then, expand according to the elements of the first column:
$$D = 2\begin{vmatrix} 0 & -3 & 23 & 11 \\ 1 & 2 & -3 & -1 \\ 0 & 5 & -7 & 4 \\ 0 & 1 & 5 & -1 \end{vmatrix} = 2\left\{-(1)\cdot\begin{vmatrix} -3 & 23 & 11 \\ 5 & -7 & 4 \\ 1 & 5 & -1 \end{vmatrix}\right\}$$
$$= 2(-598) = -1196.$$

Note 1. Recognize that **we cannot expand a determinant of order higher than 3 by a method like that used for determinants of the 3d order in Note 1 on page 61.** If, for instance, we used such a method for the deter-

minant in Problem 5, page 229, we would obtain only 8 terms, instead of the actual 24 terms of the expansion.

Note 2. The signs to use in expanding by minors, or the signs to be attached to the minors of the elements of a determinant in order to obtain the cofactors, can be remembered by use of the adjoining diagram. The sign in each place is the one which should be attached to the minor of the corresponding element in expanding. The signs alternate in proceeding to the *right* in any row, or moving *down* in any column.

$$\begin{vmatrix} + & - & + & \cdot & \cdot & \cdot & \cdot \\ - & + & - & \cdot & \cdot & \cdot & \cdot \\ + & - & + & \cdot & \cdot & \cdot & \cdot \\ - & + & - & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

EXERCISE 90

1. Expand by minors according to the elements of the 1st column; of the 2d column; of the 3d row. $\begin{vmatrix} c_1 & m_1 & v_1 \\ c_2 & m_2 & v_2 \\ c_3 & m_3 & v_3 \end{vmatrix}$.

Evaluate by use of expansion by minors.

2. $\begin{vmatrix} 1 & 5 & 2 \\ 4 & 7 & 3 \\ 2 & -3 & 6 \end{vmatrix}$. **4.** $\begin{vmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ -2 & 0 & 3 \end{vmatrix}$. **6.** $\begin{vmatrix} 3a & c & b \\ a & 2 & 3 \\ c & b & 4a \end{vmatrix}$.

3. $\begin{vmatrix} -2 & 0 & 2 \\ -1 & -3 & 1 \\ 4 & -2 & 3 \end{vmatrix}$. **5.** $\begin{vmatrix} 3 & 4 & 2 \\ -1 & 0 & -5 \\ 2 & -3 & 4 \end{vmatrix}$. **7.** $\begin{vmatrix} 1 & x & y \\ 1 & x^2 & y^2 \\ 1 & x^3 & y^3 \end{vmatrix}$.

8. $\begin{vmatrix} 6 & -5 & 1 & 4 \\ 5 & -2 & 2 & 4 \\ 7 & -8 & -2 & 4 \\ 3 & -6 & -3 & 9 \end{vmatrix}$. **11.** $\begin{vmatrix} 3 & -1 & 2 & 3 \\ 1 & 0 & 2 & 1 \\ 2 & 3 & 0 & 1 \\ 5 & 2 & 4 & -5 \end{vmatrix}$.

9. $\begin{vmatrix} 1 & 2 & 5 & 7 \\ -6 & 3 & 0 & 9 \\ -3 & 5 & 2 & 7 \\ 2 & 1 & 4 & 3 \end{vmatrix}$. **12.** $\begin{vmatrix} 3 & -1 & 4 & 2 \\ 4 & 4 & 4 & 6 \\ 8 & 5 & -8 & 4 \\ 6 & 6 & 6 & 9 \end{vmatrix}$.

10. $\begin{vmatrix} 2 & 1 & 3 & 7 \\ 3 & 4 & 5 & 8 \\ 6 & 4 & -2 & -4 \\ 8 & 4 & -3 & 3 \end{vmatrix}$. **13.** $\begin{vmatrix} 2 & 3 & 1 & 4 \\ 5 & -2 & 0 & 3 \\ 0 & 1 & 2 & -5 \\ 3 & 3 & 2 & 3 \end{vmatrix}$.

14. Without expanding, show that the adjoining equation is satisfied when $x = 2$ and $x = 3$. Hence, what factors has the determinant? Check by expanding and solving the equation. $\begin{vmatrix} 1 & 2 & 3 \\ 1 & x & 3 \\ 1 & 3 & x \end{vmatrix} = 0.$

15. Without expanding, show that the adjoining determinant has the factors $(x - y)$, $(y - w)$, and $(x - w)$, and find a factored expression for the determinant. $\begin{vmatrix} 1 & 1 & 1 \\ x & y & w \\ x^2 & y^2 & w^2 \end{vmatrix}$.

204. An auxiliary result.

ILLUSTRATION 1. Consider the expansion of the following determinant according to the minors C_1, C_2, and C_3 of the 3d column.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \equiv c_1C_1 - c_2C_2 + c_3C_3. \tag{1}$$

If, on both sides of (1), we replace c_1, c_2, and c_3 by a_1, a_2, and a_3, respectively, we obtain

$$\begin{vmatrix} a_1 & b_1 & a_1 \\ a_2 & b_2 & a_2 \\ a_3 & b_3 & a_3 \end{vmatrix} \equiv a_1C_1 - a_2C_2 + a_3C_3. \tag{2}$$

Since the determinant in (2) has two columns identical, it is identically zero. Hence, $a_1C_1 - a_2C_2 + a_3C_3 \equiv 0$. In a similar fashion, the following theorem can be proved for a determinant of any order.

THEOREM I. *In the expansion of a determinant D by minors, according to the elements of a given column, if we replace the elements of this column by the corresponding elements of another column, the resulting expression is identically zero.*

205. Solution of linear equations by determinants. Any system of n linear equations in n unknowns can be written in the form

$$\text{I.}\quad \begin{cases} a_1x + b_1y + c_1z + \cdots = k_1, \\ a_2x + b_2y + c_2z + \cdots = k_2, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_nx + b_ny + c_nz + \cdots = k_n, \end{cases}$$

where $k_1, k_2, \cdots, k_n$ are constant terms. In system I, certain of the coefficients a_1, b_1, c_2, k_1, etc. may be zeros. For (I), let the Greek capital letter delta, Δ, represent the determinant formed from the coefficients of the unknowns, *written by columns in their natural order.*

THEOREM I. *In any system of linear equations of the form* I, *containing the same number of unknowns as equations, if the determinant of the coefficients of the unknowns is not zero* ($\Delta \neq 0$), *then the system has a single solution. In this solution, the value of any unknown can be expressed as a fraction in which*

1. *the denominator is Δ, and*
2. *the numerator is the determinant obtained if, in Δ, the column of coefficients of this unknown is replaced by the column of constant terms, $k_1, k_2, \cdots, k_n$.*

Note 1. For simplicity in writing, but with general reasoning, we shall prove Theorem I for the special case of the following system:

$$\text{II.}\quad \begin{cases} a_1x + b_1y + c_1z = k_1, & (1)\\ a_2x + b_2y + c_2z = k_2, & (2)\\ a_3x + b_3y + c_3z = k_3. & (3)\end{cases}$$

In system II,
$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \qquad (4)$$

Let K_1, K_2, and K_3 be defined as follows:

$$K_1 = \begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix};\quad K_2 = \begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix};\quad K_3 = \begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}. \qquad (5)$$

The determinants K_1, K_2, and K_3 are those specified as the numerators in Step 2 of Theorem I.

Proof of Theorem I. 1. First let us show that, in any solution of (II), x, y, and z satisfy the following equations:

$$\Delta\cdot x = K_1;\qquad \Delta\cdot y = K_2;\qquad \Delta\cdot z = K_3. \qquad (6)$$

2. To establish, for instance, the second equation of (6), we proceed as follows. In Δ, let B_1, B_2, and B_3 be the minors of b_1, b_2, and b_3, respectively. On multiplying both sides of (1) by $-B_1$, of (2) by $+B_2$, and of (3) by $-B_3$, we obtain the following equations:

$$-a_1B_1x - b_1B_1y - c_1B_1z = -k_1B_1. \qquad (7)$$
$$a_2B_2x + b_2B_2y + c_2B_2z = k_2B_2. \qquad (8)$$
$$-a_3B_3x - b_3B_3y - c_3B_3z = -k_3B_3. \qquad (9)$$

On adding corresponding members of (7), (8), and (9) and collecting terms, the coefficient of y is found to be

$$-b_1B_1 + b_2B_2 - b_3B_3, \qquad (10)$$

which is seen to be Δ, expanded according to the elements of its 2d column. The coefficient of x is

$$-a_1B_1 + a_2B_2 - a_3B_3,$$

which is the same as (10) except that a's replace the b's in (10). Hence, by Theorem I, Section 204, the coefficient of x is zero. Similarly, it is found that the coefficient of z is zero. Therefore, on adding (7), (8), and (9), we obtain

$$\Delta\cdot y = -k_1B_1 + k_2B_2 - k_3B_3. \qquad (11)$$

The right member of (11) is the same as (10) except that k_1, k_2, and k_3 replace b_1, b_2, and b_3, respectively. Since (10) equals Δ, the right member of (11) is the expansion of the determinant K_2 obtained by replacing the column of b's in Δ by the constant terms k_1, k_2, and k_3. Therefore, from (11) we obtain

$$\Delta \cdot y = K_2.$$

3. Similarly, to establish $\Delta \cdot z = K_3$, we should commence by multiplying both sides of (1), (2), and (3) by $+C_1$, $-C_2$, and $+C_3$, respectively. To obtain $\Delta \cdot x = K_1$, we should commence by multiplying both sides of (1), (2), and (3) by $+A_1$, $-A_2$, and $+A_3$, respectively. Recognize that **equations 6 are true whether or not Δ is zero, if (x, y, z) is a solution of (II).**

4. **IF $\Delta \neq 0$,** from equations 6 we obtain

$$x = \frac{K_1}{\Delta}, \quad y = \frac{K_2}{\Delta}, \quad \text{and} \quad z = \frac{K_3}{\Delta}, \tag{12}$$

which completes the proof of Theorem I.

Note 2. Recognize that the equations in (6) and (12) are obtained under the assumption that *there is some set of values of* (x, y, z) which satisfy (II). Hence, the preceding proof shows that, *if there is any solution*, there is *only one*, given in (12). To complete a rigorous proof, it should be shown, by substitution in (II), that (12) actually is a solution. This substitution is omitted here.*

EXAMPLE 1. Solve:

$$\text{III.} \quad \begin{cases} 2x - y + 2z + w = 12, & (13) \\ 2x - y + 3z - 4w = 5, & (14) \\ 5x + y + z = 6, & (15) \\ -2y + z + w = 9, & (16) \end{cases}$$

SOLUTION. We use the method of Theorem I.

$$\Delta = \begin{vmatrix} 2 & -1 & 2 & 1 \\ 2 & -1 & 3 & -4 \\ 5 & 1 & 1 & 0 \\ 0 & -2 & 1 & 1 \end{vmatrix} = 48; \quad K_3 = \begin{vmatrix} 2 & -1 & 12 & 1 \\ 2 & -1 & 5 & -4 \\ 5 & 1 & 6 & 0 \\ 0 & -2 & 9 & 1 \end{vmatrix} = 144; \quad z = \frac{K_3}{\Delta} = 3.$$

Similarly, by use of determinants we find $y = -2$. The values of x and w could also be found by use of determinants, but it is more convenient to proceed as follows:

Substitute $(y = -2, z = 3)$ *in* (15) *and* (16): from (15), $x = 1$; from (16), $w = 2$. The solution of (III) is $(x = 1, y = -2, z = 3, w = 2)$.

* See L. E. DICKSON's *Elementary Theory of Equations*, page 145.

Note 3. The discussion of the solution of a general system of m linear equations in n unknowns in which $\Delta = 0$ is too complicated for treatment in this book.* Such a system may be consistent (has solutions), but usually a system is inconsistent if $\Delta = 0$. The most simple condition for inconsistency is given in the following theorem.

THEOREM II. *A system of the form* I *is inconsistent if* $\Delta = 0$, *and if any one (or more) of the numerator determinants* $K_1, K_2, K_3, \cdots$ *is different from zero.*

Proof. **IF** system II **is consistent** and if (x, y, z) represents any solution, then the equations in (6) hold, *even when* $\Delta = 0$. Since

$$\Delta \cdot x = K_1,$$

it follows that if $\Delta = 0$ then $K_1 = 0$; similarly, $K_2 = K_3 = 0$. Hence, if $\Delta = 0$ and if any one of (K_1, K_2, K_3) is *not* zero, the system *cannot* be consistent.

EXERCISE 91

Solve by determinants and check by substitution.

1. $\begin{cases} x + y - 2z = 7, \\ 2x - 3y - 2z = 0, \\ x - 2y - 3z = 3. \end{cases}$

2. $\begin{cases} 2x + 3y + 2z = 2, \\ 4x - y - z = 1, \\ 2x + y + 2z = 4. \end{cases}$

3. $\begin{cases} 6s + 3t + 2w = 1, \\ 3s - 4w = 4, \\ 5s - t = 14. \end{cases}$

4. $\begin{cases} x - 4y + 3z = 4, \\ 3x - 10y + z = 18, \\ 4x - 2y + 2z = 12. \end{cases}$

5. $\begin{cases} 2x + 4y - z + 3w = 7, \\ -2x + 2y + 3z - 2w = -6, \\ 2x - 2y + 2z + w = 9, \\ 3x - 4y - 2z + w = 8. \end{cases}$

6. $\begin{cases} 2x + 2y + 3z + w = 7, \\ x + 8y + z = 10, \\ 3x - 5z - 2w = 10, \\ 4x - y + 2z + w = 11. \end{cases}$

7. $\begin{cases} 3y + 2z + w + 3 = 0, \\ 4x + y + z + 4w = 14, \\ x - y - 4z - 7 = 0, \\ 3x - 2y - 2z + w = 12. \end{cases}$

8. $\begin{cases} 3x + y + 3z - 2w + v = 0, \\ 5 - y + 3v - w = 0, \\ z + 2v + 2 = 0, \\ 2 - x - y - z + 2w + v = 0, \\ x + y + 2z + 3w = 2. \end{cases}$

* For a complete treatment of the solution of m linear equations in n unknowns, when $m > n$, $m = n$, or $m < n$, see DICKSON, *work cited*, page 146.

$$9. \quad \begin{cases} 2x - 3y + z - u + v + 3 = 0, \\ 3y + 4z + 2u - v = 1, \\ 4x - y + z - v = -2, \\ 6x - 2y - z + u = 4, \\ 4y + 3z - u + 2v = 3. \end{cases}$$

Prove that each of the following systems is inconsistent.

$$10. \quad \begin{cases} x + 2y - 3z = 2, \\ 2x + y - 4z = 5, \\ x - y - z = 2. \end{cases} \qquad 11. \quad \begin{cases} 2x + y - z = 2, \\ 4x + 7y + z = 13, \\ x + 3y + z = 4. \end{cases}$$

206. Homogeneous equations. A linear equation is *homogeneous* in case the constant term in it is zero. Thus, the equations in (II), page 236, are homogeneous if $k_1 = k_2 = k_3 = 0$.

By substitution, we see that any system of homogeneous linear equations is satisfied if each unknown is zero. Frequently, such a solution is of no use, and hence it is sometimes called the **trivial solution.**

THEOREM I. *If a system of n homogeneous linear equations in n unknowns has a solution other than the trivial one where each unknown is zero, then* $\Delta = 0$.

Proof. Consider system II, page 236, when each of (k_1, k_2, k_3) is zero. Then, each of (K_1, K_2, K_3) is zero because each has a column of zeros. Hence, by equations 12, page 237, if $\Delta \neq 0$,

$$\left(x = \frac{K_1}{\Delta} = 0, \quad y = 0, \quad z = 0\right)$$

is the only solution. Therefore, if there *is* some solution where the unknowns are *not all zero*, it is *impossible to have* $\Delta \neq 0$.

THEOREM II. *In a system of n homogeneous linear equations in n unknowns, if* $\Delta = 0$, *then the system has infinitely many nontrivial solutions.*

Note 1. The proof of Theorem II is beyond the scope of this book. In case nontrivial solutions exist, they can usually be obtained as follows:

1. *Solve* $n - 1$ *of the equations for* $n - 1$ *of the unknowns in terms of the other unknown, — call it* x.

2. *Assign any value, not zero, to* x *and compute the values of the other unknowns by use of the results of Step* 1. *Each set of corresponding values of the n unknowns thus obtained is a solution of the system.*

EXAMPLE 1. Discuss the system: I. $\begin{cases} 3x + 2y - 3z = 0, & (1) \\ 4x - y + 7z = 0, & (2) \\ x - 3y + 10z = 0. & (3) \end{cases}$

SOLUTION. 1. First, we verify that the determinant formed from the coefficients is zero:

$$\begin{vmatrix} 3 & 2 & -3 \\ 4 & -1 & 7 \\ 1 & -3 & 10 \end{vmatrix} = 0.$$

2. Hence, by Theorem II, system I has nontrivial solutions.

3. We solve (1) and (2) for x and y in terms of z; the solution is $(x = -z, y = 3z)$. We verify that these values also satisfy (3).

4. From Step 3, if $z = 2$, then $x = -2$ and $y = 6$. Hence, one nontrivial solution of (1) is $(x = -2, y = 6, z = 2)$. Similarly, corresponding to each value of z we obtain a solution of (I). That is, (I) has infinitely many nontrivial solutions, which can be represented by the following formulas, where a may have any value: $x = -a$; $y = 3a$; $z = a$.

207. A system of linear equations containing more unknowns than equations usually has infinitely many solutions, although, under exceptional conditions, such a system may be inconsistent.

ILLUSTRATION 1. Consider the system: $A.\ \begin{cases} 3x - y - 2z = 1, \\ 2x + y - 3z = -1. \end{cases}$

In (A), the determinant of the coefficients of x and y is not zero. Hence, considering z as a constant for the moment, we can solve (A) for x and y in terms of z by use of determinants. We obtain the solution $(x = z, y = z - 1)$. From this result, if $z = 2$, then $x = 2$ and $y = 1$; that is,

$$(x = 2, \qquad y = 1, \qquad z = 2)$$

is one solution of (A). Similarly, corresponding to each value of z, we obtain a solution of (A). That is, (A) has *infinitely* many solutions, given by the following formulas where k may have any value:

$$x = k; \qquad y = k - 1; \qquad z = k.$$

208. A system of linear equations containing more equations than unknowns usually is *inconsistent*, but under certain conditions is consistent. Consider any system of m equations in n unknowns, where $m > n$, and suppose that we can solve n of the equations for the unknowns. Then, if the values obtained satisfy the other $m - n$ equations, which were not used in solving, the system is consistent. Otherwise the system is inconsistent.

ILLUSTRATION 1. Consider the adjoining system. On solving the first two equations, alone, we find that they have a single solution ($x = -3$, $y = 2$). By substitution we find that this is also a solution of the 3d equation. Hence, the system is consistent and has only one solution.

$$\begin{cases} x - 2y + 7 = 0, \\ 3x + 7y - 5 = 0, \\ x + y + 1 = 0. \end{cases}$$

A system of n linear equations in $(n-1)$ unknowns can be written as follows (illustrated for the case $n = 3$):

$$\left.\begin{aligned} a_1x + b_1y + c_1 &= 0, \\ a_2x + b_2y + c_2 &= 0, \\ a_3x + b_3y + c_3 &= 0. \end{aligned}\right\} \qquad (1)$$

THEOREM I. *In a system of n linear equations in $(n-1)$ unknowns, if the determinant formed from the coefficients and constant terms is not zero, the system is inconsistent.*

Comment. The proof is given for the special case of system 1, where $n = 3$, but the reasoning is of a general nature. It would be useful for the student to write out the proof for the case $n = 4$.

Proof. 1. Consider the following system, obtained from (1) by changing c_1 to c_1z, c_2 to c_2z, and c_3 to c_3z:

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= 0, \\ a_2x + b_2y + c_2z &= 0, \\ a_3x + b_3y + c_3z &= 0. \end{aligned}\right\} \qquad (2)$$

2. If (1) is consistent, it has a solution ($x = h$, $y = k$). Then, (2) has a solution ($x = h$, $y = k$, $z = 1$), because (2) becomes (1) if $z = 1$. In other words, if (1) is consistent, then (2) is a *homogeneous system with a nontrivial solution,* and hence, by Theorem I of Section 206, the determinant formed from the coefficients of x, y, and z in (2) is zero. Therefore, if this determinant is *not* zero, it follows that system 1 *cannot* be consistent.

EXERCISE 92

Find two nontrivial solutions, or prove that none exist.

1. $\begin{cases} 2x + 3y - z = 0, \\ 3x - y - 7z = 0, \\ x + 7y + 5z = 0. \end{cases}$

2. $\begin{cases} u - 3v - 9w = 0, \\ 2u + v - 4w = 0, \\ 3u + 5v + w = 0. \end{cases}$

3. $\begin{cases} 2x - 3y + 2z = 0, \\ 4x - 6y + 5z = 0, \\ x - y + z = 0. \end{cases}$

4. $\begin{cases} x + 3y - z = 0, \\ 2x + 6y - 2z = 0, \\ 3x - y + 2z = 0. \end{cases}$

Find two solutions of the system.

5. $\begin{cases} 6x - y + 2z = 1, \\ 4x + y - 3z = -1. \end{cases}$

6. $\begin{cases} 2x - 3y - z + 1 = 0, \\ x + 2y + 3z = 3. \end{cases}$

Prove inconsistent, or find a solution.

7. $\begin{cases} x - 2y + 7 = 0, \\ x + 11y - 19 = 0, \\ 3x + 7y = 5. \end{cases}$

9. $\begin{cases} 3x - 5y + 3 = 0, \\ x - 2y + 5 = 0, \\ 3x + 5y + 2 = 0. \end{cases}$

8. $\begin{cases} x - 2y = 3, \\ 3x - 5y - 10 = 0, \\ 2x - 3y - 7 = 0. \end{cases}$

10. $\begin{cases} x - 2y + 5 = 0, \\ 2x - 4y + 7 = 0, \\ x + 3y - 2 = 0. \end{cases}$

Given that the system is consistent, find the value of the constant k.

11. $\begin{cases} x + 2ky + 1 = 0, \\ 3kx + 14y - 5 = 0, \\ 2x + 5ky + 1 = 0. \end{cases}$

12. $\begin{cases} x + 2y + k = 0, \\ 3kx - ky - 1 = 0, \\ x - y + 1 = 0. \end{cases}$

Given that nontrivial solutions exist, find the value of the constant k.

13. $\begin{cases} 2kx - y + 4kz = 0, \\ 8x + 5y + kz = 0, \\ 2x + y + z = 0. \end{cases}$

14. $\begin{cases} 3kx - y + 2w = 0, \\ 2y + kw = 0, \\ kx + 3y - w = 0. \end{cases}$

15. If (x, y, z) is a solution of the system, show that $x : y : z = 1 : -4 : -11$.
$$\begin{cases} 3x + z = 2y, \\ 2x + 2z = 5y. \end{cases}$$

16. If (x, y, z) is any solution of the adjoining system, show that $x : y : z = 1 : 2 : 1$.
$$\begin{cases} -2x + 3y - 4z = 0, \\ 3x - y - z = 0. \end{cases}$$

17. If (x, y, z) is a solution of the adjoining system, and if at least one of the determinants which appear below is not zero, prove that
$$\begin{cases} a_1x + b_1y + c_1z = 0, \\ a_2x + b_2y + c_2z = 0. \end{cases}$$

$$x : y : z = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} : \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} : \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

CHAPTER EIGHTEEN

Partial Fractions

209. Introduction. If $f(x)$ and $g(x)$ are polynomials, then $f(x)/g(x)$ is called a **rational fraction.** A rational fraction is called a **proper fraction** if the degree of the numerator is less than the degree of the denominator. In advanced mathematics it is sometimes necessary to decompose a given proper fraction into a sum of more simple fractions, called **partial fractions.** The determination of these partial fractions is based on the following theorem, whose proof is beyond the scope of this book.

THEOREM I. *A proper fraction N/D, which is in its lowest terms, can be resolved into a sum of partial fractions made up as follows:*

I. *If a linear factor $ax + b$ occurs once as a factor of D, there is a partial fraction $\dfrac{A}{ax+b}$, in which A is some constant, not zero.*

II. *If $ax + b$ occurs k times as a factor of D, there are k partial fractions*

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k},$$

where $A_1, A_2, \cdots, A_k$ are constants and $A_k \neq 0$.

III. *If a quadratic factor $lx^2 + mx + n$ occurs once in D, there is a partial fraction $\dfrac{Cx+F}{lx^2+mx+n}$, where C and F are constants, not both zero.*

IV. *If $lx^2 + mx + n$ occurs k times as a factor of D, there are k partial fractions,*

$$\frac{C_1x+F_1}{lx^2+mx+n} + \frac{C_2x+F_2}{(lx^2+mx+n)^2} + \cdots + \frac{C_kx+F_k}{(lx^2+mx+n)^k},$$

in which the C's and F's are constants, and $C_kx + F_k \not\equiv 0$.

In this chapter, we shall be concerned with fractions $f(x)/g(x)$ where $f(x)$ and $g(x)$ have real coefficients, and shall be interested

only in partial fractions with real coefficients. Hence, by a *linear factor* we shall mean one with *real coefficients;* by a *quadratic factor,* we shall mean a *quadratic factor which has no real linear factors.*

An **improper fraction** is one where the degree of the numerator is greater than or equal to the degree of the denominator. To decompose an improper fraction, we first reduce it to a sum of a polynomial and a proper fraction. Then, the resulting proper fraction can be decomposed by applying Theorem I.

ILLUSTRATION 1. $$\frac{x^3+3x^2-5}{x^2+5} = x+3-\frac{5x+20}{x^2+5}.$$

210. Case I. *When the denominator can be expressed as a product of real linear factors, all of which are different.*

EXAMPLE 1. Resolve into partial fractions:

$$\frac{7x^2-23x+10}{(3x-1)(x-1)(x+2)}.$$

SOLUTION. 1. We shall find constants A, B, and C such that, for all values of x except those for which some denominators are zero,

$$\frac{7x^2-23x+10}{(3x-1)(x-1)(x+2)} = \frac{A}{3x-1}+\frac{B}{x-1}+\frac{C}{x+2}. \qquad (1)$$

2. On clearing (1) of fractions we obtain

$$7x^2-23x+10$$
$$= A(x-1)(x+2)+B(3x-1)(x+2)+C(3x-1)(x-1). \qquad (2)$$

3. Equation 2 is true for all values of x except possibly $x=1$, $x=-2$, and $x=\frac{1}{3}$, for each of which some denominators in (1) are zero, and thus (2) is true for infinitely many values of x. Hence, it follows from Corollary 1, page 140, that (2) is true for *all* values of x, *including even* $x=1$, $x=-2$, *and* $x=\frac{1}{3}$.

4. From (2),

when $x=1$: $\quad -6 = A\cdot 0+6B+C\cdot 0;\quad B=-1.$

when $x=-2$: $\quad 84 = 21C;\quad C=4.$

when $x=\frac{1}{3}$: $\quad \frac{28}{9} = -\frac{14}{9}A;\quad A=-2.$

5. Therefore, on substituting in (1), we obtain

$$\frac{7x^2-23x+10}{(3x-1)(x-1)(x+2)} = -\frac{2}{3x-1}-\frac{1}{x-1}+\frac{4}{x+2}. \qquad (3)$$

CHECK. On reducing to a common denominator, and adding the fractions in the right member of (3), we obtain the left member.

211. Case II. *When the factors of the denominator are of the first degree, and some repeated.*

Example 1. Resolve into partial fractions: $\dfrac{4x^3 + 16x^2 - 5x + 3}{(x - 1)^2(x + 2)^2}$.

First solution. 1. We shall find constants A, B, C, and D so that, for all values of x except those for which some denominators are zero,

$$\frac{4x^3 + 16x^2 - 5x + 3}{(x - 1)^2(x + 2)^2} = \frac{A}{(x - 1)^2} + \frac{B}{x - 1} + \frac{C}{(x + 2)^2} + \frac{D}{x + 2}. \qquad (1)$$

2. On clearing of fractions in (1), we obtain

$$4x^3 + 16x^2 - 5x + 3 \qquad (2)$$

$$= A(x + 2)^2 + B(x - 1)(x + 2)^2 + C(x - 1)^2 + D(x + 2)(x - 1)^2 \qquad (3)$$

$$= (B + D)x^3 + (A + 3B + C)x^2$$
$$+ (4A - 2C - 3D)x + (4A - 4B + C + 2D). \qquad (4)$$

In (4), we expanded (3) and collected corresponding powers of x.

3. Since (2) = (4) for infinitely many values of x, the coefficient of each power of x in (4) equals the coefficient of that power in (2). Hence,

$$\left.\begin{aligned} B + D &= 4,\\ A + 3B + C &= 16,\\ 4A - 2C - 3D &= -5,\\ 4A - 4B + C + 2D &= 3. \end{aligned}\right\} \qquad (5)$$

4. On solving (5) for A, B, C, and D, we obtain $A = 2$, $B = 3$, $C = 5$, and $D = 1$. Hence,

$$\frac{4x^3 + 16x^2 - 5x + 3}{(x - 1)^2(x + 2)^2} = \frac{2}{(x - 1)^2} + \frac{3}{x - 1} + \frac{5}{(x + 2)^2} + \frac{1}{x + 2}.$$

Second solution. 1. Place $x = 1$ in (2) and (3):

$$18 = 9A; \qquad A = 2.$$

2. Place $x = -2$ in (2) and (3):

$$45 = 9C; \qquad C = 5.$$

3. Place $A = 2$ and $C = 5$ in the first two equations of (5):

$$B + D = 4; \qquad 3B = 9. \qquad (6)$$

4. On solving (6), we obtain $B = 3$ and $D = 1$. In this solution, we would not even write down the last two equations in (5), or the corresponding terms in (4).

To check any expansion into partial fractions, combine the result into a single fraction to obtain the original fraction.

EXERCISE 93

1. $\dfrac{5x+16}{(x-4)(2x+1)}$.

2. $\dfrac{13x+18}{(3x+2)(x-4)}$.

3. $\dfrac{2x-5}{x^2-5x-14}$.

4. $\dfrac{7x-14}{x^2-3x-10}$.

5. $\dfrac{4y^2+14y+18}{y(y+3)^2}$.

6. $\dfrac{x^2-4x+3}{x(x+1)^2}$.

7. $\dfrac{x^2+1}{(x-1)^3}$.

8. $\dfrac{3a^2+10a+9}{(a+2)^3}$.

9. $\dfrac{5x^2-11x+5}{(x-1)(x^2-3x+2)}$.

10. $\dfrac{x^3-3x^2+2x-2}{(x^2-x)(x-1)^2}$.

11. $\dfrac{x^3-7x+7}{(x-2)(x^2-3x+2)}$.

12. $\dfrac{x^4-4x^3+x+27}{x(x-3)^2}$.

212. Case III. *When the denominator contains quadratic factors which are not repeated.*

Example 1. Resolve into partial fractions: $\dfrac{3x^2-x+1}{(x+1)(x^2-x+3)}$.

Solution. 1. We shall find constants A, B, and C such that, for all values of x except those for which some denominators are zero,

$$\frac{3x^2-x+1}{(x+1)(x^2-x+3)}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+3}. \tag{1}$$

2. Clear of fractions in (1):

$$3x^2-x+1=A(x^2-x+3)+(Bx+C)(x+1), \tag{2}$$

or,

$$3x^2-x+1=(A+B)x^2+(B+C-A)x+(3A+C). \tag{3}$$

3. Place $x=-1$ in (2): $5=5A$; $A=1$.

4. Equate coefficients of x^2 and of x in (3):

$$A+B=3; \qquad B+C-A=-1. \tag{4}$$

On using $A=1$ in (4), we obtain $B=2$ and $C=-2$. Hence,

$$\frac{3x^2-x+1}{(x+1)(x^2-x+3)}=\frac{1}{x+1}+\frac{2x-2}{x^2-x+3}.$$

213. Case IV.* *When the denominator contains quadratic factors, some of which are repeated.*

Example 1. Resolve into partial fractions:

$$\frac{x^4+x^3+2x^2-7}{(x+2)(x^2+x+1)^2}.$$

* The following Exercise 94 is arranged so that Case IV may be omitted.

SOLUTION. 1. We shall find constants A, B, C, D, and E so that

$$\frac{x^4 + x^3 + 2x^2 - 7}{(x + 2)(x^2 + x + 1)^2} = \frac{A}{x + 2} + \frac{Bx + C}{(x^2 + x + 1)^2} + \frac{Dx + E}{x^2 + x + 1}. \tag{1}$$

2. Clear of fractions in (1) and collect terms on the right:

$$x^4 + x^3 + 2x^2 - 7 \tag{2}$$

$$= A(x^2 + x + 1)^2 + (Bx + C)(x + 2) + (Dx + E)(x^2 + x + 1)(x + 2) \tag{3}$$

$$= (A + D)x^4 + (2A + E + 3D)x^3 + (3A + B + 3D + 3E)x^2 + (2A + 2B + C + 2D + 3E)x + (A + 2C + 2E). \tag{4}$$

3. On placing $x = -2$ in (2) and (3), one obtains $9A = 9;\ A = 1$.

4. Equate the coefficients of x^4, x^3, x^2, and x in (2) and (4):

$$\left.\begin{aligned} A + D &= 1,\\ 2A + E + 3D &= 1,\\ 3A + B + 3D + 3E &= 2,\\ 2A + 2B + C + 2D + 3E &= 0. \end{aligned}\right\} \tag{5}$$

5. In (5), insert $A = 1$, and solve for B, C, D, and E. The results are

$$B = 2; \qquad C = -3; \qquad D = 0; \qquad E = -1.$$

$$\therefore \quad \frac{x^4 + x^3 + 2x^2 - 7}{(x + 2)(x^2 + x + 1)^2} = \frac{1}{x + 2} + \frac{2x - 3}{(x^2 + x + 1)^2} - \frac{1}{x^2 + x + 1}.$$

Comment. Frequently, useful equations connecting the unknown constants can be easily obtained by placing x equal to convenient values other than those which make the linear factors of the denominator D equal to zero. Thus, in Example 1, besides placing $x = -2$ in (2) and (3), we could place $x = 0$; this gives $-7 = A + 2C + 2E$ which is the same as would be obtained by equating the constant terms in (2) and (4). With this equation as a substitute, we could then omit the last equation of (5).

Note 1. Recall Corollary 1, page 141; *any polynomial $g(x)$ with real coefficients can be expressed as a product of real linear and real quadratic factors.* Hence, this chapter furnishes a complete treatment of the problem of decomposing a fraction $f(x)/g(x)$ with real coefficients into its partial fractions, if we assume that we can solve $g(x) = 0$.

EXERCISE 94

Resolve into partial fractions.

1. $\dfrac{2x^2 + 11x - 7}{(2x - 5)(x^2 + 2)}$.

2. $\dfrac{z^2 - 7z + 8}{(z + 2)(2z^2 + 5)}$.

3. $\dfrac{4x^2 + 3x + 14}{x^3 - 8}$.

4. $\dfrac{4x^3 + 4x^2 + 54x + 18}{x^4 - 81}$.

5. $\dfrac{2x^3 + 8x^2 + 2x + 4}{(x+1)^2(x^2+3)}$.

6. $\dfrac{5x^2 + 3x + 6}{x(x^2 - x + 3)}$.

7. $\dfrac{8x^3 + x^2 + 19x + 5}{(2x^2+3)(x^2+5)}$.

8. $\dfrac{7x^3 + 2x^2 + 13x + 2}{4x^4 + 11x^2 + 7}$.

9. $\dfrac{3x^2 - 5x + 6}{(1+3x)(1 - 2x - 15x^2)}$.

10. $\dfrac{2x^3 + 3x^2 - 3x + 4}{(x-1)(x^3-1)}$.

11. $\dfrac{2x}{8x^3 + 1}$.

12. $\dfrac{9x^2 + 14x + 3}{2x^3 + 5x^2 + 3x}$.

13. $\dfrac{2x^4 + 3x^3 + 2x^2 - 2x + 1}{x^3(x^2+1)}$.

14. $\dfrac{6x^4 - 10x^3 + 18x^2 - 14x}{(x^2-1)(x^2+2x-3)}$.

15. $\dfrac{4x^3 - 3x^2 - 20}{(x^2+x+4)(2x^2+2x+5)}$.

16. $\dfrac{3x^3 + 4x^2 + 12x - 17}{(2x^2+3x+5)(3x-x^2-4)}$.

Note 1. Some of the following problems involve Case IV.

17. $\dfrac{x^3 + x^2 + 2}{(x^2+2)^2}$.

18. $\dfrac{2x^3 + x + 3}{(x^2+1)^2}$.

19. $\dfrac{5x^2 + 18x - 2}{(x^2+3x+1)^2}$.

20. $\dfrac{10x^2 + 8x + 27}{(2x^2+x+5)^2}$.

21. $\dfrac{6x^3 + 11x^2 - 1}{(x-1)^2(2x^2+x+1)^2}$.

22. $\dfrac{x^7 + x^5 + x^3 + x}{(x^2+2)^2(x^2+3)^2}$.

23. $\dfrac{17 + 52x + 30x^2 - 24x^3}{(3x^2 - x - 2)^2}$.

24. $\dfrac{x^4 - 4x - 13}{(x^2+3)(x^2+x+2)^2}$.

25. $\dfrac{81 - 155x + 123x^2 - 24x^3}{(2x^2 - 7x + 3)^2}$.

CHAPTER NINETEEN

Infinite Series

214. Limit of a sequence. Let $S_1, S_2, S_3, \cdots, S_n, \cdots$ be an endless sequence of numbers.* Then, we introduce the following fundamental notion.

DEFINITION I. *To say that the limit of S_n as n becomes infinite is S means that, if any positive number d is assigned, then there is a corresponding place in the sequence such that, for all terms S_n beyond this place, the absolute value of the difference $S - S_n$ is less than d.*

To abbreviate "*the limit of S_n as n becomes infinite is S*," we write

$$\operatorname*{limit}_{n\to\infty} S_n = S \quad \text{or} \quad \text{limit}_{n\to\infty} S_n = S.$$

ILLUSTRATION 1. Consider the sequence $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \cdots$, where $S_n = 1 - \dfrac{1}{2^n}\cdot$ If n is sufficiently large, $\dfrac{1}{2^n}$ will be as close to zero as we please, and hence S_n will be as close to 1 as we please. Therefore, $\operatorname*{limit}_{n\to\infty} S_n = 1$. This is exhibited geometrically below where the points $S_1 = \frac{1}{2}$, $S_2 = \frac{3}{4}$, etc., approach to practical coincidence with $S = 1$.

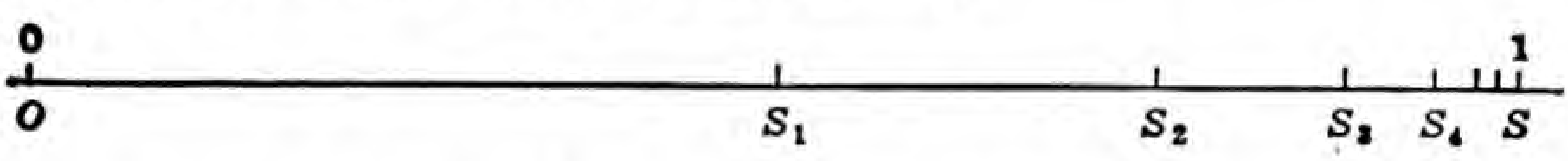


ILLUSTRATION 2. A simple illustration of a variable is one which is a constant, that is, one whose values are all equal. If $S_n = k$ for all values of n, then $S_n - k = 0$ for every n, and hence

$$\operatorname*{limit}_{n\to\infty} S_n = k.$$

Note 1. The definition of "$\operatorname*{limit}_{n\to\infty} S_n = S$" could not be construed as *preventing S_n from being equal to its limit S*, or as *requiring S_n to equal S for some value or values of n*. In Illustration 1, S_n is *never* equal to its limit. In Illustration 2, S_n is *always* equal to its limit. For the sequence $1, \frac{1}{2}, 1, \frac{3}{4}, 1, \frac{7}{8}, \cdots$, the limit is 1, and there are infinitely many terms equal to 1 as well as infinitely many different from 1.

* All numbers in this chapter will be real. However, the stated definition of a limit applies to the case where S_n is a complex number.

ILLUSTRATION 3. If $S_n = 2 + (-1)^n \frac{1}{2^n}$, then

$$\lim_{n\to\infty} S_n = 2.$$

In this case, $S_1 = \frac{3}{2}$; $S_2 = 2\frac{1}{4}$; $S_3 = 1\frac{7}{8}$; $S_4 = 2\frac{1}{16}$; etc. It is seen that the terms of the sequence are alternately less than and greater than 2.

215. Theorems on limits. In the following theorems, we refer to variables S_n and T_n where

$$\operatorname*{limit}_{n\to\infty} S_n = S; \qquad \operatorname*{limit}_{n\to\infty} T_n = T.$$

I. *The limit of the sum of two variables is the sum of the limits of the variables. That is,*

$$\operatorname*{limit}_{n\to\infty} (S_n + T_n) = \operatorname*{limit}_{n\to\infty} S_n + \operatorname*{limit}_{n\to\infty} T_n = S + T.$$

II. *The limit of the product of two variables is the product of the limits of the variables. That is,*

$$\operatorname*{limit}_{n\to\infty} S_nT_n = (\operatorname*{limit}_{n\to\infty} S_n)(\operatorname*{limit}_{n\to\infty} T_n) = ST.$$

In particular, if k is any constant, then $\operatorname*{limit}_{n\to\infty} kS_n = kS$.

ILLUSTRATION 1. If $\operatorname*{limit}_{n\to\infty} S_n = 5$, and $\operatorname*{limit}_{n\to\infty} T_n = -3$, then

$$\operatorname*{limit}_{n\to\infty} (S_n + T_n) = 5 - 3 = 2; \qquad \operatorname*{limit}_{n\to\infty} S_nT_n = 5(-3) = -15.$$

III. *The limit of the quotient of two variables is the quotient of the limits of the variables, provided that* **the limit of the denominator is not zero.** *That is, if*

$$\operatorname*{limit}_{n\to\infty} T_n \neq 0, \quad \text{then} \quad \operatorname*{limit}_{n\to\infty} \frac{S_n}{T_n} = \frac{\operatorname*{limit}_{n\to\infty} S_n}{\operatorname*{limit}_{n\to\infty} T_n} = \frac{S}{T}.$$

ILLUSTRATION 2. If $\operatorname*{limit}_{n\to\infty} S_n = 6$, and $\operatorname*{limit}_{n\to\infty} T_n = 5$, then $\operatorname*{limit}_{n\to\infty} \frac{S_n}{T_n} = \frac{6}{5}$.

Note 1. Theorem III makes no assertion about the limit of a quotient *when the denominator has zero as its limit.* Such a quotient may, or may not, approach a limit, depending on the behavior of the numerator.

Note 2. The proofs of Theorems I, II, and III are not difficult, but require mature appreciation of the definition of a limit. These theorems will be used whenever necessary, without formal reference. Proofs of the theorems are given in advanced calculus.

216. Infinite series. If $u_1, u_2, u_3, \cdots, u_n, \cdots$ is an endless sequence of terms, the expression

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots \qquad (1)$$

is called an *infinite series.* Hereafter, the word *series* will refer to an *infinite series.*

Let $S_1 = u_1$; $S_2 = u_1 + u_2$; in general, let S_n represent the sum of the first n terms in (1), or

$$S_n = u_1 + u_2 + \cdots + u_n.$$

We call S_n a *partial sum* for the infinite series. Then, we define the notions of *convergence* and *sum* for an infinite series as follows.

Definition I. *An infinite series is said to* **converge,** *if the sum of the first n terms approaches a limit as n becomes infinite. If S is this limit, that is, if* $\lim_{n\to\infty} S_n = S$, *then we call S the* **sum** *of the series, and say that it* **converges** *to S.*

Definition II. *If a series does not converge, we say that it* **diverges.**

Illustration 1. For the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots$, we obtain $S_1 = \frac{1}{2}$; $S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$; $S_3 = \frac{7}{8}$; etc. In general, by use of formula 5, page 107, for a geometric progression,

$$S_n = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}; \qquad \lim_{n\to\infty} S_n = 1 - 0 = 1.$$

Hence, the given series converges to the sum 1. In this case, all the partial sums are less than the sum of the series.

Illustration 2. For the series $2 + 4 + \cdots + 2n + \cdots$, we find $S_1 = 2$, $S_2 = 6$, $S_3 = 12$. By the formula for the sum of an arithmetic progression, $S_n = 2 + 4 + \cdots + 2n = n(1 + n)$. If n grows large without bound, then S_n becomes infinite and therefore does not approach any limit. Hence, the given series diverges.

Illustration 3. For the series $1 - 1 + 1 - 1 + \cdots$, we obtain $S_1 = 1$; $S_2 = 1 - 1 = 0$; $S_3 = 1$; etc. Hence, as n becomes infinite, S_n does not approach a limit because the terms $S_1, S_2, S_3, \cdots$ oscillate between 0 and 1. Therefore, the given series diverges.

Note 1. The sum of an infinite series is *not a sum* in the ordinary sense of the word, but is the limit of the sequence of partial sums $S_1, S_2, S_3, \cdots$. In defining the sum of a series, we have not introduced some supernatural way of adding infinitely many terms.

217. Necessary condition for convergence. *If a series converges, then the* nth *term approaches zero as n becomes infinite.*

Proof. Let $u_1 + u_2 + \cdots$ be the series, and let S be its sum. Then, $S_n = S_{n-1} + u_n$; or, $u_n = S_n - S_{n-1}$. Since the series converges to S, both S_n and S_{n-1} approach S as $n \to \infty$. Hence,

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} (S_n - S_{n-1}) = S - S = 0.$$

This necessary condition establishes the following result.

THEOREM I. *If the* nth *term of a series does not approach zero as n becomes infinite, then the series diverges.*

ILLUSTRATION 1. The series $1 + 2 + \cdots + n + \cdots$ diverges, because $u_n = n$, which does *not* approach zero as $n \to \infty$. On the other hand, the nth term in $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ *does* approach zero but, as will be proved later, this series also diverges.

218. To determine an approximation to the sum S of a convergent series when no convenient formula for S is available, we recall that the partial sum S_n will be as close to S as we may wish, if n is sufficiently large. Usually, we must decide by intuition how large n must be in order for S_n to be as close to S as we desire. Occasionally, it is possible to determine an upper limit of the error made in accepting any specified partial sum S_n as an approximation to S.

ILLUSTRATION 1. Let us obtain the sum of the following convergent series, correct to four decimal places:

$$1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots + \frac{1}{(2n-2)!} + \cdots.$$

$$\begin{array}{llll}
1 = 1 & = 1.00000. & S_1 = 1 & = 1.00000 \\
\frac{1}{2!} = \frac{1}{2} & = .50000. & S_2 = 1 + \frac{1}{2!} & = 1.50000 \\
\frac{1}{4!} = \frac{1}{24} & = .04167. & S_3 = 1 + \frac{1}{2!} + \frac{1}{4!} & = 1.54167 \\
\frac{1}{6!} = \frac{1}{720} & = .00139. & S_4 = 1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} & = 1.54306 \\
\frac{1}{8!} = \frac{1}{40320} & = .00002. & S_5 = 1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \frac{1}{8!} & = 1.54308
\end{array}$$

It is reasonably certain that the terms beyond $\frac{1}{8!}$ will be so small that the sum of any number of them will not affect any partial sum S_n by more than

.00001. Hence, we are reasonably certain that, to four decimal places, the sum of the series is $S = 1.5431$.

Note 1. In more advanced mathematics, it is proved that the following infinite series converge and give the values of the corresponding functions, for *all* values of x in the case of (1), (2), and (3), and for all values of x such that $|x| < 1$ in the case of (4).

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^{n+1}\frac{x^{2n-2}}{(2n-2)!} + \cdots. \tag{1}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{n+1}\frac{x^{2n-1}}{(2n-1)!} + \cdots. \tag{2}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \cdots. \tag{3}$$

$$\arcsin x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{x^7}{7} + \cdots. \tag{4}$$

In (1) and (2), x is in radian measure. In (3), $e = 2.71828\cdots$, which is the base of the natural system of logarithms. We call (3) the **exponential series.** In (4), arcsin x represents the *principal* value of the function.

THEOREM I. *The infinite geometric series* $a + ar + ar^2 + ar^3 + \cdots$ *converges if* $|r| < 1$, *and diverges if* $|r| \geqq 1$.

Proof. In Section 112 it was proved that the series converges if $|r| < 1$, with $a/(1-r)$ as its sum. By Theorem I, Section 217, the series diverges if $|r| \geqq 1$, because in this case ar^{n-1} does *not* approach zero as $n \to \infty$.

219. The binomial series. By the binomial theorem, for a positive integral exponent, n,

$$(y + x)^n = y^n + ny^{n-1}x + \frac{n(n-1)}{1\cdot 2}y^{n-2}x^2 + \cdots [\textit{to } (n+1) \textit{ terms}]. \tag{1}$$

The following related infinite series is called the *binomial series:*

$$1 + kx + \frac{k(k-1)}{1\cdot 2}x^2 + \frac{k(k-1)(k-2)}{1\cdot 2\cdot 3}x^3 + \cdots. \tag{2}$$

Note 1. If k is not a positive integer, (2) contains infinitely many non-vanishing terms. If n is a positive integer, and if we place $k = n$ in (2), the series contains only $(n + 1)$ terms which are not zero, because, beyond the $(n + 1)$th term, each numerator has zero as one factor. Moreover, these $(n + 1)$ terms are exactly those obtained if in (1) we place $y = 1$. That is, *if k is a positive integer, series 2 reduces to the expansion of* $(1 + x)^k$.

Thus, if $k = 3$ in (2), we obtain

$$1 + 3x + 3x^2 + \frac{3(3-1)(3-2)}{1\cdot 2\cdot 3}x^3 + \frac{3(3-1)(3-2)(3-3)}{1\cdot 2\cdot 3\cdot 4}x^4 + \cdots,$$

or simply $1 + 3x + 3x^2 + x^3 + 0 + 0 + \cdots$, which is $(1 + x)^3$.

If k is any real number, the first part of the following theorem can be proved by the methods of advanced mathematics; the second part will be proved later in this chapter.

THEOREM I. *If x has any value for which the binomial series* 2 *converges, then its sum is $(1 + x)^k$. Moreover, for all values of k,* (2) *converges if $|x| < 1$ and diverges if $|x| > 1$.*

EXAMPLE 1. By use of (2), find $\sqrt[3]{1.1}$ to four decimal places.

SOLUTION. $\sqrt[3]{1.1} = (1 + .1)^{\frac{1}{3}}$. We substitute $x = .1$ and $k = \frac{1}{3}$ in (2) and determine the sum of the series:

$$\sqrt[3]{1.1} = 1 + .03333 - .00111 + .00006 - \cdots = 1.0323.$$

Note 2. Let a and b represent any two numbers, and suppose that $|b| > |a|$. Then, $(b + a)^k$ can be expanded by use of (2):

$$(b + a)^k = b^k\left(1 + \frac{a}{b}\right)^k = b^k\left[1 + k\frac{a}{b} + \frac{k(k-1)}{2!}\left(\frac{a}{b}\right)^2 + \cdots\right]; \tag{3}$$

$$(b + a)^k = b^k + kb^{k-1}a + \frac{k(k-1)}{2!}b^{k-2}a^2 + \cdots, \tag{4}$$

which has the same form as (1), except that (4) is an infinite series when k is not an integer. In (3), we employed (2) with $x = \dfrac{a}{b}$; this value of x satisfies the condition $|x| < 1$ under which (2) converges to $(1 + x)^k$.

EXERCISE 95

It will be proved later that each of the following series converges. Compute the sum of each series using enough terms to obtain the result correct to three decimal places.

1. $1 + \frac{1}{2^5} + \frac{1}{3^5} + \cdots.$ **2.** $\frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots.$

Note 1. It will be proved later that, in an *alternating* series such as those in Problems 3 and 4, any partial sum S_n differs numerically from S, the sum of the series, by *at most the absolute value of the first omitted term.* This fact is useful in deciding when enough terms have been used in Problems 3 and 4.

3. $1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \cdots.$ **4.** $\frac{1}{1\cdot 2^3} - \frac{1}{2\cdot 3^3} + \frac{1}{3\cdot 4^3} - \frac{1}{4\cdot 5^3} + \cdots.$

Compute correct to two decimal places by use of a series from page 253.

5. $\sqrt{e}$. **6.** e. **7.** $\sin \frac{1}{2}$. **8.** $\cos .2$. **9.** $\sin .1$. **10.** $\arcsin \frac{1}{2}$.

Compute correct to two decimal places; use the binomial series.

11. $(1 + .1)^{\frac{1}{3}}$. **13.** $(1.01)^{-3}$. **15.** $\sqrt[3]{1.04}$. **17.** $\sqrt{110}$. **19.** $\sqrt[3]{12}$.
12. $(1.05)^{\frac{1}{4}}$. **14.** $(1.02)^{-5}$. **16.** $\sqrt{39}$. **18.** $\sqrt[3]{120}$. **20.** $\sqrt[4]{24}$.

HINT for Problem 16. Express 39 as the nearest perfect square plus a suitable number: $\sqrt{39} = \sqrt{36 + 3}$. Then, use Note 2, page 254.

Obtain the following formulas by use of the binomial series.

21. $\sqrt{1 - x} = 1 - \frac{1}{2}x - \frac{1}{2 \cdot 4}x^2 - \frac{1 \cdot 3}{2 \cdot 4 \cdot 6}x^3 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8}x^4 + \cdots.$

22. $\dfrac{1}{1 + x} = 1 - x + x^2 - x^3 + \cdots.$

23. $\dfrac{1}{(1 + x)^2} = 1 - 2x + 3x^2 - 4x^3 + \cdots.$

24. $\dfrac{1}{\sqrt{1 + x}} = 1 - \frac{1}{2}x + \frac{1 \cdot 3}{2 \cdot 4}x^2 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \cdots.$

★*Prove the following theorems by use of the definition of convergence.*

25. If $u_1 + u_2 + \cdots$ converges to the sum S, and if $v_1 + v_2 + \cdots$ converges to the sum T, then $(u_1 + v_1) + (u_2 + v_2) + (u_3 + v_3) + \cdots$ converges and its sum is $(S + T)$.

26. If $u_1 + u_2 + \cdots$ converges to the sum S, then $au_1 + au_2 + \cdots$ converges to the sum aS; and, if $u_1 + u_2 + \cdots$ diverges, then $au_1 + au_2 + \cdots$ diverges, for every value of $a \neq 0$.

CONVERGENCE TESTS FOR SERIES OF POSITIVE TERMS

220. Fundamental assumption. *Let S_1, S_2, $\cdots$, S_n, $\cdots$ be an endless sequence of real numbers in which S_n* **never decreases** *as n becomes infinite, and* **always remains less than some fixed number A.** *Then, S_n has a limit as n becomes infinite and this limit is not greater than A. That is, if*

$$S_n \leqq S_{n+1} \quad \textit{and} \quad S_n \leqq A$$

for all values of n, then there exists a number $S \leqq A$ such that $\text{limit}_{n\to\infty} S_n = S$.

Note 1. Even in an advanced treatment of the theory of limits, it is sometimes found convenient to adopt the preceding assumption as a basis rather than to employ other fundamental postulates which lead to the proof of the assumption as a theorem. We had an instance of the truth of the assumption in Illustration 1, page 249.

THEOREM I. *If $u_1 + u_2 + \cdots$ is an infinite series of positive terms, and if there exists a number A such that, for all values of n, the sum of n terms, S_n, is not greater than A, then the series converges and its sum S is not greater than A.*

Proof. By definition, $S_{n+1} = S_n + u_{n+1}$. Since all terms u_1, u_2, $\cdots$ are positive, hence $S_n < S_{n+1}$. Moreover, by hypothesis, $S_n \leqq A$ for all values of n. Therefore, by the fundamental assumption, the sequence $S_1, S_2, \cdots$ has a limit S, and $S \leqq A$.

EXERCISE 96

1. Test the truth of the fundamental assumption as in Illustration 1, page 249, if $S_n = 2 - n^{-1}$.

2. (*a*) State the analogue of the fundamental convergence assumption for $S_1, S_2, \cdots, S_n, \cdots$, where S_n never increases as n increases. (*b*) State and prove the analogue of Theorem I for a series where all terms are negative.

221. Comparison tests. Consider two series of nonnegative terms:

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots. \qquad (U)$$

$$v_1 + v_2 + v_3 + \cdots + v_n + \cdots. \qquad (V)$$

THEOREM I. *If (V) converges and if each term of (U) is less than or equal to the corresponding term of (V), then (U) converges.*

Proof. 1. Let $S_n = u_1 + u_2 + \cdots + u_n$; $T_n = v_1 + v_2 + \cdots + v_n$. Since $u_1 \leqq v_1$, $u_2 \leqq v_2$, etc., hence $S_n \leqq T_n$ for all values of n.

2. Let T be the sum of (V); then $\text{limit}_{n\to\infty} T_n = T$. Since $v_n \geqq 0$, T_n does not decrease as n increases and hence $T_n \leqq T$ for all n.

3. Since $S_n \leqq T_n$, hence $S_n \leqq T$ for all values of n. Therefore, by Theorem I, Section 220, (U) converges to a sum at most equal to T.

THEOREM II. *If (U) diverges and if each term of (V) is greater than or equal to the corresponding term of (U), then (V) diverges.*

Proof. Since $u_n \leqq v_n$, if (V) *were* convergent, then from Theorem I it would follow that (U) *converges.* But, this contradicts our hypothesis and hence (V) *cannot* converge, or therefore (V) *diverges.*

EXAMPLE 1. Prove convergent:

$$1 + \frac{1}{2^2} + \frac{1}{3^3} + \cdots + \frac{1}{n^n} + \cdots. \qquad (1)$$

SOLUTION. The infinite series

$$1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} + \cdots \qquad (2)$$

converges because it is a geometric series whose ratio is $\frac{1}{2}$. We see that

$$n^n = 2^{n-1} \quad if \quad n = 1; \quad n^n > 2^{n-1} \quad if \quad n \geqq 2,$$

because then n^n is the product of n factors each *at least* 2, while 2^{n-1} is the product of $(n - 1)$ factors 2. Hence, $n^n \geqq 2^{n-1}$ for all values of n and each term of (1) is less than or equal to the corresponding term of (2). Therefore, by Theorem I, the given series converges.

222. Comparison series. Any series of positive terms which is known to converge, or to diverge, is eligible as a comparison series for proving the convergence or divergence of other series. The following series are particularly useful. We discussed (1) and (5) in Theorem I, Section 218, and (3) in Section 221.

Convergent series for comparison:

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots \ (a > 0;\ 0 \leqq r < 1). \tag{1}$$

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots \ (p > 1). \tag{2}$$

$$1 + \frac{1}{2^2} + \frac{1}{3^3} + \cdots + \frac{1}{n^n} + \cdots. \tag{3}$$

Divergent series for comparison:

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots. \tag{4}$$

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots \ (a > 0;\ r \geqq 1). \tag{5}$$

Proof that (4) *diverges.* 1. Consider the inequalities

$$1 + \frac{1}{2} > 1,$$

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2},$$

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2},$$

$$\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16} > \frac{1}{16} + \frac{1}{16} + \cdots + \frac{1}{16} = \frac{1}{2},$$

$$\dots\dots \text{etc.} \dots\dots$$

2. On adding corresponding sides of the preceding inequalities, including all down to a certain level, we obtain for S_n, the sum of n terms,

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > 1 + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2}, \tag{6}$$

where n depends on how many inequalities were included. The right side in (6) can be made as large as we please, by including enough inequalities; or, the sum S_n grows large without limit as $n \to \infty$. Therefore, (4) diverges.

★*Comment.* The kth inequality in Step 1 includes 2^{k-1} terms on the left whose denominators range from $(2^{k-1}+1)$ to 2^k, and the sum of these terms would be greater than $2^{k-1}\cdot\frac{1}{2^k}$, or $\frac{1}{2}$, for all values of $k \geqq 2$.

We call (4) the **harmonic series** because its terms form a harmonic progression.

Proof that (2) *converges.* 1. Consider the inequalities

$$\left.\begin{aligned}
&1 \leqq 1,\\
&\frac{1}{2^p}+\frac{1}{3^p} \leqq \frac{1}{2^p}+\frac{1}{2^p}=\frac{2}{2^p}=\frac{1}{2^{p-1}},\\
&\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p} \leqq \frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}=\frac{4}{4^p}=\frac{1}{4^{p-1}}=\left(\frac{1}{2^{p-1}}\right)^2,\\
&\frac{1}{8^p}+\frac{1}{9^p}+\cdots+\frac{1}{15^p} \leqq \frac{8}{8^p}=\frac{1}{8^{p-1}}=\left(\frac{1}{2^3}\right)^{p-1}=\left(\frac{1}{2^{p-1}}\right)^3,\\
&\qquad\qquad \ldots\ldots\ldots\ldots \text{etc.} \ldots\ldots\ldots\ldots
\end{aligned}\right\} \tag{7}$$

2. The sum of the terms on the right in (7) is the geometric series

$$1+\frac{1}{2^{p-1}}+\left(\frac{1}{2^{p-1}}\right)^2+\cdots \qquad \left(\textit{whose ratio is } r=\frac{1}{2^{p-1}}\right). \tag{8}$$

Since $p > 1$, we have $r < 1$; hence, (8) converges. If A is the sum of (8), it follows that the sum of any number of terms at the left of the "$\leqq$" signs in (7) will be at most A. Therefore, since all the terms of (2) are present on the left in (7), we have $S_n \leqq A$ for every value of n, where S_n is a partial sum for (2). Hence, by Theorem I, Section 220, series 2 converges.

★*Comment.* The kth inequality in (7) includes 2^{k-1} terms on the left, with denominators ranging from $(2^{k-1})^p$ to $(2^k-1)^p$ inclusive. The sum of these terms is at most $\dfrac{2^{k-1}}{(2^{k-1})^p}=\dfrac{1}{(2^{k-1})^{p-1}}=\dfrac{1}{(2^{p-1})^{k-1}}$.

The following theorem is useful in applying the comparison tests.

THEOREM I. *The convergence or divergence of a series is not affected by adding on (or dropping out) a finite number of terms.*

Proof. 1. Drop out a group of terms from $u_1+u_2+\cdots$ whose sum is A. Let S_n be the sum of n terms of the original series and T_k the sum of k terms of the new series.

2. For any value of k, T_k can be obtained by subtracting A from some partial sum S_n for the original series. That is, $T_k = S_n - A$. Therefore, if S is the sum of the original series, then $S - A$ is the sum of the new series, because

$$\lim_{k\to\infty} T_k = \lim_{n\to\infty} (S_n - A) = S - A.$$

3. Similarly, if the T-series converges, then the S-series converges. Hence, either both series converge or both diverge.

ILLUSTRATION 1. To prove that the series

$$\sqrt{3} + \sqrt{2} + 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots \tag{9}$$

diverges, we drop off the first two terms, obtaining

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} + \cdots. \tag{10}$$

By comparison with (4), we see that (10) diverges, because, for all values of n, we have $\dfrac{1}{\sqrt{n}} \geqq \dfrac{1}{n}\cdot$ Hence, by Theorem I, (9) also diverges.

EXERCISE 97

Prove convergent by use of the comparison test. The series in any problem may be used as a comparison series in any later problem.

1. $\dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 2^2} + \dfrac{1}{3\cdot 2^3} + \cdots.$

2. $\dfrac{1}{2\cdot 1^2} + \dfrac{1}{3\cdot 2^2} + \dfrac{1}{4\cdot 3^2} + \cdots.$

3. $\dfrac{1}{1+3^1} + \dfrac{1}{1+3^2} + \cdots.$

4. $\dfrac{1}{1+1^2} + \dfrac{1}{1+2^2} + \cdots.$

5. $\dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots.$

6. $\dfrac{1}{2^p} + \dfrac{1}{4^p} + \dfrac{1}{6^p} + \cdots \; (p > 1).$

7. $\dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \dfrac{1}{3\cdot 4} + \cdots.$

8. $\dfrac{1}{1!} + \dfrac{1}{2!} + \dfrac{1}{3!} + \cdots.$

9. $\dfrac{1}{1!} + \dfrac{1}{3!} + \dfrac{1}{5!} + \cdots.$

10. $\dfrac{1}{2!} + \dfrac{1}{4!} + \dfrac{1}{6!} + \cdots.$

Prove divergent by use of the comparison test or Section 217.

11. $3 + 5 + 7 + \cdots.$

12. $\dfrac{1}{1^{\frac{1}{4}}} + \dfrac{1}{2^{\frac{1}{4}}} + \dfrac{1}{3^{\frac{1}{4}}} + \cdots.$

13. $\dfrac{3}{1} + \dfrac{3^2}{2} + \dfrac{3^3}{3} + \cdots.$

14. $\dfrac{2}{1} + \dfrac{3}{2} + \dfrac{4}{3} + \cdots.$

15. $\dfrac{1}{1^p} + \dfrac{1}{2^p} + \dfrac{1}{3^p} + \cdots \; (p < 1).$

16. $\dfrac{1}{1-\frac{1}{2}} + \dfrac{1}{1-\frac{1}{3}} + \dfrac{1}{1-\frac{1}{4}} + \cdots.$

Prove divergent, by the method used for (4), *on page* 257.

17. $1 + \frac{1}{3} + \frac{1}{5} + \cdots.$

18. $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots.$

19. Prove in two ways that $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots$ diverges.

Establish the convergence or the divergence of each series.

20. $1 + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}} + \cdots.$

21. $\frac{1}{1\cdot 3} + \frac{1}{2\cdot 4} + \frac{1}{3\cdot 5} + \cdots.$

22. $\frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \frac{1}{3\cdot 4\cdot 5} + \cdots.$

23. $\frac{1}{1-\sqrt{.1}} + \frac{1}{2-\sqrt{.02}} + \frac{1}{3-\sqrt{.003}} + \cdots.$

24. $\frac{2}{1}\cdot\frac{1}{1^2} + \frac{3}{2}\cdot\frac{1}{2^2} + \frac{4}{3}\cdot\frac{1}{3^2} + \cdots.$

25. $\frac{2}{3}\cdot\frac{1}{1} + \frac{3}{4}\cdot\frac{1}{2} + \frac{4}{5}\cdot\frac{1}{3} + \cdots.$

26. $\frac{3}{2\cdot 1} + \frac{4}{3\cdot 2} + \frac{5}{4\cdot 3} + \cdots.$

27. $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots.$

28. $\frac{2}{1\cdot 3} + \frac{2\cdot 4}{1\cdot 3\cdot 5} + \frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5\cdot 7} + \cdots.$

★**29.** Let $u_1 + u_2 + \cdots$ and $v_1 + v_2 + \cdots$ be series of positive terms, and suppose that there exists a constant k such that $u_n \leqq kv_n$ for all values of n. Then, if the v-series converges, prove that the u-series converges.

★**30.** State and prove a generalized comparison test for divergence similar to the result of Problem 29.

★*Prove convergent or divergent by use of Problems* 29 *and* 30.

31. $\frac{2}{1}\cdot\frac{1}{3} + \frac{3}{2}\cdot\frac{1}{3^2} + \frac{4}{3}\cdot\frac{1}{3^3} + \cdots.$

32. $\frac{2}{3}\cdot\frac{1}{1} + \frac{3}{4}\cdot\frac{1}{2} + \frac{4}{5}\cdot\frac{1}{3} + \cdots.$

223. Ratio test. Consider a series of positive terms,

$$u_1 + u_2 + u_3 + \cdots + u_n + u_{n+1} + \cdots, \tag{1}$$

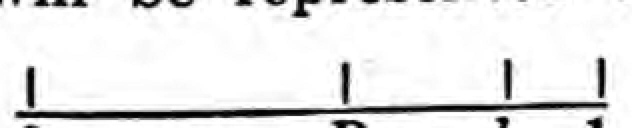
and let $r_n = \frac{u_{n+1}}{u_n}$; we shall call r_n the **test ratio.**

THEOREM I. *Suppose that*

$$\underset{n\to\infty}{limit} \frac{u_{n+1}}{u_n} = R. \tag{2}$$

Then, (A) if $R < 1$, the series converges; (B) if $R > 1$, the series diverges; (C) if $R = 1$, the test fails and the series may converge or it may diverge.

Proof of (A). 1. Since $R < 1$, we may select a number h between R and 1. Since $limit_{n\to\infty}\, r_n = R$, if n is sufficiently large then r_n will be represented on the following scale by a point to



$(R < h < 1)$

the *left* of h, and $r_n < h$. Consequently, if the integer k is chosen

sufficiently large, and if $n \geqq k$, then

$$\frac{u_{n+1}}{u_n} < h. \tag{3}$$

Place $n = k$ in (3): $\quad \dfrac{u_{k+1}}{u_k} < h, \quad$ or $\quad u_{k+1} < hu_k.$

Place $n = k + 1$ in (3): $\quad \dfrac{u_{k+2}}{u_{k+1}} < h, \quad$ or $\quad u_{k+2} < hu_{k+1} < h^2u_k.$

Place $n = k + 2$ in (3): $\quad \dfrac{u_{k+3}}{u_{k+2}} < h, \quad$ or $\quad u_{k+3} < hu_{k+2} < h^3u_k.$

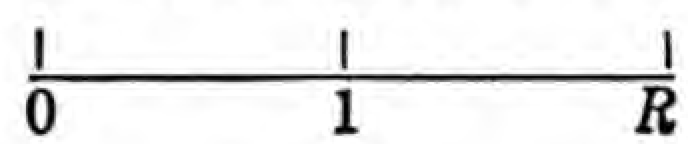
. . . .

Thus, each term in the following series 4 is less than the corresponding term in series 5.

$$u_{k+1} + u_{k+2} + u_{k+3} + \cdots; \tag{4}$$
$$hu_k + h^2u_k + h^3u_k + \cdots. \tag{5}$$

2. Series 5 *converges*, because it is a geometric series whose ratio is h, which is *less than* 1. Therefore, by comparison, (4) converges. But, (4) is obtained by dropping out $u_1 + u_2 + \cdots + u_k$ from (1); hence, by Theorem I, page 258, (1) converges.

Proof of (*B*). 1. Since $R > 1$ and since $\mathit{limit}_{n\to\infty}\, r_n = R$, if n is sufficiently large then r_n will be represented on the following scale by a point not farther to the left than the unit point, 1, and $1 \leqq r_n$.



That is, if the integer k is chosen sufficiently large and if $n \geqq k$, then $r_n \geqq 1$, or

$$\frac{u_{n+1}}{u_n} \geqq 1. \tag{6}$$

2. On placing $n = k$, then $n = k + 1$, etc., in (6), we obtain

$$u_{k+1} \geqq u_k; \qquad u_{k+2} \geqq u_{k+1}; \qquad u_{k+3} \geqq u_{k+2}; \; \cdots;$$

or, each term beyond u_k is *at least as large as the preceding term*. Hence, we cannot have $\mathit{limit}_{n\to\infty}\, u_n = 0$, and, therefore, by Theorem I, page 252, it follows that the original series does not converge.

Proof of (*C*). If $R = 1$, we can draw no conclusion about the convergence of the given infinite series because, as will be seen later, there are both convergent and divergent series for which $R = 1$.

ILLUSTRATION 1. By later methods, it can be shown that $R = 1$ for the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$, which diverges, and also that $R = 1$ for the series $1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \cdots$, which converges.

To abbreviate the statement "r_n *becomes infinite as* n *becomes infinite*," we write

$$\operatorname*{limit}_{n\to\infty} r_n = \infty.$$

In this case, although r_n has *no limit* in the normal sense as $n \to \infty$, nevertheless (6) is true if n is sufficiently large and hence (1) *diverges.* For convenience, we may think of this case as being included under (B) for $R = \infty$.

Note 1. Even if r_n has *no limit,* or if $\operatorname{limit}_{n\to\infty} r_n = 1$, nevertheless if $r_n \geqq 1$ for all values of n then (1) diverges, because (6) is a suitable foundation for the proof of (B). This modification of (B) is sometimes useful.

ILLUSTRATION 2. In the series $\dfrac{1}{2} + \dfrac{2^2}{2^2} + \dfrac{3^2}{2^3} + \cdots$,

$$u_n = \frac{n^2}{2^n}; \qquad u_{n+1} = \frac{(n+1)^2}{2^{n+1}}; \qquad \frac{u_{n+1}}{u_n} = \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2}.$$

Therefore, $$R = \operatorname*{limit}_{n\to\infty} \frac{2^n(n+1)^2}{2^{n+1}\cdot n^2} = \operatorname*{limit}_{n\to\infty} \frac{(n+1)^2}{2n^2} = \operatorname*{limit}_{n\to\infty} \frac{n^2+2n+1}{2n^2}.$$

To evaluate the limit, divide both numerator and denominator by n^2, the *highest power of* n in them, and apply (III) and then (I) of page 250:

$$R = \operatorname*{limit}_{n\to\infty} \frac{1 + \dfrac{2}{n} + \dfrac{1}{n^2}}{2} = \frac{1+0+0}{2} = \frac{1}{2}.$$

Hence, since $R < 1$, the series converges.

Comment. We could also find R in Illustration 2 as follows:

$$R = \operatorname*{limit}_{n\to\infty} \frac{(n+1)^2}{2n^2} = \frac{1}{2} \operatorname*{limit}_{n\to\infty} \left(\frac{n+1}{n}\right)^2 = \frac{1}{2} \operatorname*{limit}_{n\to\infty} \left(1 + \frac{1}{n}\right)^2 = \frac{1}{2}\cdot(1+0)^2 = \frac{1}{2}.$$

ILLUSTRATION 3. In the series $\dfrac{1}{3} + \dfrac{2!}{3^2} + \dfrac{3!}{3^3} + \cdots$, we have

$$u_n = \frac{n!}{3^n}; \qquad u_{n+1} = \frac{(n+1)!}{3^{n+1}}; \qquad \frac{u_{n+1}}{u_n} = \frac{(n+1)!}{3^{n+1}} \cdot \frac{3^n}{n!} = \frac{(n+1)!}{3\cdot n!}.$$

$$R = \operatorname*{limit}_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots n\cdot(n+1)}{1\cdot 2\cdot 3\cdots n} \cdot \frac{1}{3} = \operatorname*{limit}_{n\to\infty} \frac{n+1}{3} = \infty.$$

Since $R > 1$, the given series diverges.

ILLUSTRATION 4. In the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$,

$$u_n = \frac{1}{n}; \quad u_{n+1} = \frac{1}{n+1}; \quad \frac{u_{n+1}}{u_n} = \frac{1}{n+1} \cdot \frac{n}{1} = \frac{n}{n+1}.$$

On dividing both numerator and denominator by n, we obtain

$$R = \underset{n\to\infty}{limit} \frac{1}{1 + \frac{1}{n}} = \frac{1}{1+0} = 1.$$

Since $R = 1$, the ratio test gives no information concerning the harmonic series, although (as proved previously) it diverges.

EXERCISE 98

Find the limit of the fraction as n becomes infinite.

1. $\dfrac{5 - 3n^2 + 6n^3}{2n^3 + 1}$.

2. $\dfrac{(n+1)!}{3n(n!)}$.

3. $\dfrac{1 + 2^n}{1 + 2^{n+1}}$.

Test for convergence by the ratio test: if the test fails, apply some other test or state a conclusion from previous knowledge.

4. $1 + \frac{1}{3} + \frac{1}{3^2} + \cdots$.

5. $\frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3} + \cdots$.

6. $2 + \frac{2^2}{2} + \frac{2^3}{3} + \cdots$.

7. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots$.

8. $\frac{3}{1^2} + \frac{3^2}{2^2} + \frac{3^3}{3^2} + \cdots$.

9. $2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \cdots$.

10. $\frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots$.

11. $\frac{1^2}{5} + \frac{2^2}{5^2} + \frac{3^2}{5^3} + \cdots$.

12. $1 + \frac{1}{3!} + \frac{1}{5!} + \cdots$.

13. $\frac{1!}{2} + \frac{2!}{2^2} + \frac{3!}{2^3} + \cdots$.

14. $\frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \cdots$.

15. $\frac{2}{1\cdot 2} + \frac{2^2}{2\cdot 3} + \frac{2^3}{3\cdot 4} + \cdots$.

16. $1 + \frac{2}{3!} + \frac{3}{5!} + \cdots$.

17. $1 + \frac{\sqrt{2}}{3!} + \frac{\sqrt{3}}{5!} + \cdots$.

18. $\frac{1!}{1^2} + \frac{2!}{2^2} + \frac{3!}{3^2} + \cdots$.

19. $\frac{1\cdot 2}{3} + \frac{2\cdot 3}{3^2} + \frac{3\cdot 4}{3^3} + \cdots$.

20. $\frac{1}{1 + 1^2} + \frac{1}{2 + 2^2} + \cdots$.

21. $\frac{1!}{1 + 1^2} + \frac{2!}{2 + 2^2} + \cdots$.

22. $\frac{1}{\sqrt{2} - 1} + \frac{1}{\sqrt{3} - 1} + \cdots$.

23. $\frac{1 + \sqrt{1}}{1!} + \frac{2 + \sqrt{2}}{2!} + \cdots$.

24. $\frac{1!}{1\cdot 2 + \sqrt{1}} + \frac{2!}{3\cdot 4 + \sqrt{2}} + \frac{3!}{5\cdot 6 + \sqrt{3}} + \cdots$.

25. $\frac{1}{(2\cdot 3)\cdot 5}+\frac{3}{(4\cdot 5)\cdot 5^2}+\frac{5}{(6\cdot 7)\cdot 5^3}+\frac{7}{(8\cdot 9)\cdot 5^4}+\cdots.$

★26. Prove Part B of the ratio test by introducing a divergent comparison series in powers of h and a procedure like the proof given for Part A.

CONVERGENCE TESTS FOR SERIES WITH POSITIVE AND NEGATIVE TERMS

224. Series of negative terms. If all terms are negative in a series, we settle the question of its convergence or divergence by considering a new series of positive terms formed by changing the signs of the given terms. If S is the sum of the new series, then $-S$ is the sum of the original series.

225. Alternating series. An *alternating series* is one whose terms are alternately positive and negative.

THEOREM I. *An alternating series converges if the absolute value of each term is less than that of the preceding one and if the nth term approaches zero, as n becomes infinite.*

Proof. 1. Let the series be $u_1 - u_2 + u_3 - \cdots$, where u_1, u_2, u_3, $\cdots$ are positive and $u_1 > u_2 > u_3 > \cdots$. If $2k$ is any even positive integer, then

$$S_{2k} = (u_1 - u_2) + (u_3 - u_4) + \cdots + (u_{2k-1} - u_{2k}); \qquad (1)$$

$$S_{2k} = u_1 - (u_2 - u_3) - (u_4 - u_5) - \cdots - u_{2k}. \qquad (2)$$

Since $u_1 > u_2$, hence, $u_1 - u_2 > 0$. Similarly, in (1) and (2) each difference in parentheses is positive. Thus, we see from (1) that S_{2k} continually increases as k increases and, from (2), that $S_{2k} < u_1$ for all values of k. Therefore, by the convergence assumption of page 255, S_{2k} approaches a limit S as k becomes infinite.

2. Consider the sum of an odd number of terms:

$$S_{2k+1} = S_{2k} + u_{2k+1}.$$

By hypothesis, $limit_{k\to\infty}\, u_{2k+1} = 0$. We have already proved that $limit_{k\to\infty}\, S_{2k} = S$. Hence,

$$\underset{k\to\infty}{limit}\, S_{2k+1} = \underset{k\to\infty}{limit}\,(S_{2k} + u_{2k+1}) = S + 0 = S.$$

Since the partial sums S_n of *even* index, $n = 2k$, and also those of *odd* index, $n = 2k + 1$, approach S as a limit when $k\to\infty$, it follows that, without restriction as to the character of n, we have $limit_{n\to\infty}\, S_n = S$. Hence, the series converges.

Note 1. It is left as an exercise for the student to prove Theorem I for a series $-u_1 + u_2 - \cdots$, whose first term is *negative.*

COROLLARY 1. *Under the hypotheses of Theorem* I, *the absolute value of the error made in accepting a partial sum* S_n *as an approximation to* S *is at most* u_{n+1}. *That is,*

$$|S - S_n| \leqq u_{n+1}.$$

Proof. Since S_n is the sum of the first n terms,

$$|S - S_n| = u_{n+1} - u_{n+2} + u_{n+3} - \cdots;$$

$$|S - S_n| = u_{n+1} - (u_{n+2} - u_{n+3}) - (u_{n+4} - u_{n+5}) - \cdots \leqq u_{n+1}. \quad (3)$$

The "$\leqq$" in (3) is a consequence of the same variety of reasoning as was employed in (2).

SUMMARY. *To apply the test of Theorem* I, *it must be shown that*

$$u_n > u_{n+1} \quad \text{and that} \quad \lim_{n\to\infty} u_n = 0.$$

ILLUSTRATION 1. In the series $1 - \dfrac{1}{2!} + \dfrac{1}{3!} - \cdots$, we have

$$u_n = \frac{1}{n!}; \qquad u_{n+1} = \frac{1}{(n+1)!}; \qquad \frac{1}{n!} > \frac{1}{(n+1)!}.$$

Hence, $u_n > u_{n+1}$ because $n + 1 > n$ in the denominators, and

$$\text{limit}_{n\to\infty}\, u_n = 0;$$

therefore the given alternating series converges.

226. Absolute convergence. A series

$$u_1 + u_2 + \cdots + u_n + \cdots \quad (1)$$

is said to be **absolutely convergent,** or to **converge absolutely,** if the series

$$|u_1| + |u_2| + \cdots + |u_n| + \cdots, \quad (2)$$

formed by taking the *absolute values of all terms,* is convergent. The notion of absolute convergence is important because of the following result.

THEOREM I. *A series converges if the series formed from the absolute values of the terms converges. That is, if a series is absolutely convergent then it converges in the ordinary sense.*

Proof. 1. By hypothesis, (2) converges; we wish to show that (1) converges. Let the sum of n terms in (1) be S_n, and in (2) be T_n.

2. In S_n, let P_n be the sum of the *positive terms* and Q_n be the *sum of the absolute values of the negative terms.* Then,

$$S_n = P_n - Q_n \quad \text{and} \quad T_n = P_n + Q_n. \tag{3}$$

3. Since (2) converges, it has a sum T, or $\lim_{n\to\infty} T_n = T$. Since all terms in (2) are positive, $T_n \leqq T$. Hence, from (3), $P_n + Q_n \leqq T$ and therefore

$$P_n \leqq T \quad \text{and} \quad Q_n \leqq T. \tag{4}$$

Moreover, as n increases, P_n and Q_n *never decrease.* Therefore, by the fundamental convergence assumption of page 255, P_n and Q_n approach limits P and Q respectively as n becomes infinite.

4. Since $S_n = P_n - Q_n$, therefore

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (P_n - Q_n) = P - Q.$$

Hence, $u_1 + u_2 + u_3 + \cdots$ converges, and its sum is $P - Q$.

The preceding Steps 3 and 4 also establish the following result.

COROLLARY 1. *If a series converges absolutely, then, separately, the series formed by the positive terms converges to a sum P and the series formed by the negative terms converges to a sum Q, and the sum of the given series is $P - Q$.*

ILLUSTRATION 1. By the ratio test, we proved that the series

$$1 + \frac{1}{2!} + \frac{1}{3!} + \cdots \tag{5}$$

converges. Therefore, by Theorem I, the alternating series

$$1 - \frac{1}{2!} + \frac{1}{3!} - \cdots \tag{6}$$

converges absolutely, and hence converges in the ordinary sense, because (5) is the series formed by the absolute values of the terms in (6).

ILLUSTRATION 2. By the test for alternating series, we find that the alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \cdots$ *converges.* But, the series of absolute values of the terms, $1 + \frac{1}{2} + \frac{1}{3} + \cdots$, does *not* converge. Hence, the alternating harmonic series *converges,* although *it does not converge absolutely.* Such a series is said to be **conditionally convergent.** In an advanced treatment of infinite series, the distinction between absolutely convergent and conditionally convergent series becomes important.

227. General ratio test. *Let $u_1 + u_2 + \cdots$ be any series of real valued terms, and suppose that*

$$\underset{n\to\infty}{\text{limit}} \left| \frac{u_{n+1}}{u_n} \right| = R.$$

(A) If $R < 1$, the series converges. (B) If $R > 1$, the series diverges. (C) If $R = 1$, no conclusion can be drawn.

Proof of (*A*). Notice that $\left| \frac{u_{n+1}}{u_n} \right|$ is the test ratio for the series $|u_1| + |u_2| + \cdots$, where all terms are positive. Since $R < 1$, this series converges, by the ratio test for a series of positive terms. Hence, the given series *converges*, because it is *absolutely convergent.*

Proof of (*B*). Since $R > 1$, it follows that, for all values of n sufficiently large,

$$\left| \frac{u_{n+1}}{u_n} \right| > 1, \quad \text{or} \quad |u_{n+1}| > |u_n|.$$

That is, beyond a certain place in the series, each term is larger, numerically, than the preceding term; hence, we do *not* have $\text{limit}_{n\to\infty}\, u_n = 0$. Therefore, by Section 217, the original series diverges.

To prove Part *C*, we could repeat the remarks concerning Part *C* of the ratio test for series in which all terms are positive.

ILLUSTRATION 1. In the series $\frac{2}{1!} - \frac{2^2}{2!} + \frac{2^2}{3!} - \cdots$,

$$|u_n| = \frac{2^n}{n!}; \qquad |u_{n+1}| = \frac{2^{n+1}}{(n+1)!}; \qquad \left| \frac{u_{n+1}}{u_n} \right| = \frac{2}{n+1}.$$

we have $\underset{n\to\infty}{\text{limit}} \frac{2}{n+1} = 0$; hence the series converges.

SUMMARY OF TESTS FOR CONVERGENCE.

1. *If u_n does not approach zero as $n\to\infty$, the series diverges.*
2. *The test for convergence applying to alternating series.*
3. *The ratio test, in the form of Section 227.*
4. *The comparison test for series of positive terms.*

If the ratio test fails to apply to a series containing both positive and negative terms, we then consider the series of absolute values of the terms and attempt to apply the comparison test. If the new series converges, the given series converges absolutely.

228. Power series. If $c_0, c_1, c_2, \cdots$ are constants, then

$$c_0 + c_1x + c_2x^2 + \cdots + c_{n-1}x^{n-1} + c_nx^n + \cdots \tag{1}$$

is called a *power series* in x. Any series of type 1 converges if $x = 0$. A power series in x may converge for *all* values of x, or for *no* values except $x = 0$, or the series may converge for *some* values of x besides $x = 0$ and diverge for other values. The ratio test is used in finding the values of x for which a power series in x converges.

EXAMPLE 1. Test for convergence:

$$1 - \frac{x^2}{2\cdot 2^2} + \frac{x^4}{3\cdot 2^4} - \cdots.$$

SOLUTION. 1. In order to apply Section 227, we obtain

$$|u_n| = \frac{x^{2n-2}}{n\cdot 2^{2n-2}}; \qquad |u_{n+1}| = \frac{x^{2n}}{(n+1)\cdot 2^{2n}};$$

$$\underset{n\to\infty}{limit}\left|\frac{u_{n+1}}{u_n}\right| = \underset{n\to\infty}{limit}\left|\frac{x^2}{4}\cdot\frac{n}{n+1}\right| = \frac{x^2}{4}\cdot 1 = \frac{x^2}{4}.$$

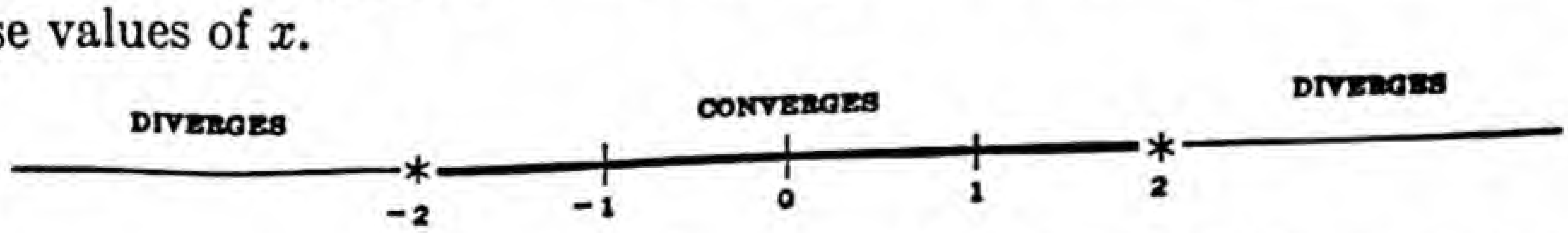
2. By the ratio test, the series

converges if $\quad \dfrac{x^2}{4} < 1; \quad$ *or* $\quad x^2 < 4; \quad$ *or* $\quad |x| < 2;$

diverges if $\quad \dfrac{x^2}{4} > 1; \quad$ *or* $\quad x^2 > 4; \quad$ *or* $\quad |x| > 2.$

3. *The ratio test fails if* $\quad \dfrac{x^2}{4} = 1, \quad$ *or* $\quad x = \pm 2.$

4. *Test for* $x = \pm 2$: the series becomes $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$, which converges, by the test for alternating series. Hence, the original series converges if $-2 \leqq x \leqq 2$ and diverges if $x < -2$ or if $x > 2$. This final conclusion is represented graphically in the following diagram. We place a star at $x = 2$ and at $x = -2$ to emphasize that the series converges for these values of x.



Note 1. For any power series in x, by advanced methods it can be proved that, when the values of x for which the series converges are represented graphically, these values form an *interval* (with zero at its center), called the **interval of convergence** of the series. A series may or may not converge for the value of x at either end point of this interval. In investigating the convergence of a power series by means of the ratio test, ***the end values of any interval of convergence must be tested separately.***

EXERCISE 99

Test the alternating series for convergence, using Section 225 if it applies.

1. $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$

2. $\frac{1}{1\cdot 2} - \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} - \cdots.$

3. $\frac{1}{1^2+1} - \frac{1}{2^2+1} + \frac{1}{3^2+1} - \cdots.$

4. $\frac{1}{\sqrt{1}+2} - \frac{1}{\sqrt{2}+3} + \frac{1}{\sqrt{3}+4} - \cdots.$

5. $\frac{1}{2\cdot 4} - \frac{1\cdot 3}{2\cdot 4\cdot 6} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 8} - \cdots.$

6. $\frac{1^2}{1!} - \frac{2^2}{2!} + \frac{3^2}{3!} - \cdots.$

7. $\frac{1!}{1^2} - \frac{2!}{2^2} + \frac{3!}{3^2} - \cdots.$

8. $\frac{1}{1.1} - \frac{1}{1.11} + \frac{1}{1.111} + \cdots.$

9. $\frac{2!}{5^1} - \frac{4!}{5^2} + \frac{6!}{5^3} + \cdots.$

Find the interval of convergence, giving a separate discussion for the ends of the interval, and represent the results graphically.

10. $1 + \frac{x}{2^2} + \frac{x^2}{3^2} + \cdots.$

11. $1 + \frac{x}{\sqrt{2}} + \frac{x^2}{\sqrt{3}} + \cdots.$

12. $1 + 2x + 2^2x^2 + \cdots.$

13. $1 + \frac{x}{4} + \frac{x^2}{4^2} + \cdots.$

14. $1 - \frac{x}{3} + \frac{x^2}{9} + \cdots.$

15. $1 + \frac{x^2}{2} + \frac{x^4}{4} + \cdots.$

16. $1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots.$

17. $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots.$

18. $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots.$

19. $\frac{2}{1}\cdot\frac{x^2}{2^2} + \frac{3}{2}\cdot\frac{x^4}{2^4} + \frac{4}{3}\cdot\frac{x^6}{2^6} + \cdots.$

20. $\frac{1}{2}\cdot\frac{x}{1} + \frac{1}{3}\cdot\frac{x^2}{2} + \frac{1}{4}\cdot\frac{x^3}{3} + \cdots.$

21. $1 + \frac{1}{2^4}\cdot\frac{x}{3} + \frac{1}{3^4}\cdot\frac{x^2}{3^2} + \cdots.$

22. $1! + x(2!) + x^2(3!) + \cdots.$

23. $1 + 2^2x + 3^2x^2 + \cdots.$

24. $\frac{2}{1}\cdot\frac{x^3}{3} + \frac{2}{1}\cdot\frac{4}{3}\cdot\frac{x^5}{5} + \frac{2}{1}\cdot\frac{4}{3}\cdot\frac{6}{5}\cdot\frac{x^7}{7} + \cdots.$

25. $\frac{1\cdot 3}{2\cdot 4}\cdot\frac{1}{2} + \frac{3\cdot 5}{4\cdot 6}\cdot\frac{x^2}{4} + \frac{5\cdot 7}{6\cdot 8}\cdot\frac{x^4}{6} + \frac{7\cdot 9}{8\cdot 10}\cdot\frac{x^6}{8} + \cdots.$

26. $\frac{1}{1\cdot 2}\cdot\frac{x^2}{1^3} + \frac{1}{3\cdot 4}\cdot\frac{x^4}{3^3} + \frac{1}{5\cdot 6}\cdot\frac{x^6}{5^3} + \cdots.$

27. $1 + \frac{x-2}{3\cdot 2^2} + \frac{(x-2)^2}{3^2\cdot 3^2} + \frac{(x-2)^3}{3^3\cdot 4^2} + \cdots.$

28. Prove that the binomial series for $(1+x)^k$ converges if $|x| < 1$, for all values of k.

CHAPTER TWENTY

Method of Least Squares and Related Topics

229. Minimum value of a quadratic function. *If p, s, and t are constants* and $p > 0$, the function $pz^2 + sz + t$ has its least value if and only if $z = -\ s/2p$.*

Proof. Let $f(z) = pz^2 + sz + t$. Then, on adding and also subtracting $s^2/4p$ to complete a square, we find

$$f(z) = \left(pz^2 + sz + \frac{s^2}{4p}\right) + t - \frac{s^2}{4p}$$

$$= p\left(z^2 + \frac{s}{p}z + \frac{s^2}{4p^2}\right) + t - \frac{s^2}{4p}.$$

Hence,
$$f(z) = p\left(z + \frac{s}{2p}\right)^2 + t - \frac{s^2}{4p}.$$

Since $p\left(z + \dfrac{s}{2p}\right)^2 \geqq 0$ for all values of z, we see that $f(z)$ has its *least* value when and only when

$$z + \frac{s}{2p} = 0, \quad \text{or} \quad z = -\frac{s}{2p}.$$

ILLUSTRATION 1. Let $f(x) = 3x^2 - 2x - 7$. The minimum value of $f(x)$ is attained when

$$x = -\frac{-2}{6} = \frac{1}{3}.$$

Hence, on the parabola which is the graph of $f(x)$, the abscissa of the vertex, or lowest point, is $x = \frac{1}{3}$.

230. The **arithmetic mean** of n numbers $(x_1, x_2, \cdots, x_n)$ is defined as *the sum of the numbers divided by n.* If A represents this arithmetic mean,

$$A = \frac{x_1 + x_2 + \cdots + x_n}{n}. \tag{1}$$

* In this chapter, all numbers will be real.

ILLUSTRATION 1. The arithmetic mean of (2, 5, 7, 16) is

$$A = \tfrac{1}{4}(2 + 5 + 7 + 16) = 7\tfrac{1}{2}.$$

If x and z are any numbers, we shall call $x - z$ the **deviation** of x from z.

THEOREM I. *The sum of the squares of the deviations of the n numbers* $(x_1, x_2, \cdots, x_n)$ *from their arithmetic mean is less than the sum of the squares of their deviations from any other number.*

Proof. 1. Let z be any number and let

$$S(z) = (x_1 - z)^2 + (x_2 - z)^2 + \cdots + (x_n - z)^2; \tag{2}$$

$S(z)$ is the sum of the squares of the deviations of the x's from z. We wish to show that $S(z)$ has its *least* value if $z = A$.

2. On expanding in (2), we obtain

$$S(z) = (x_1^2 - 2x_1z + z^2) + (x_2^2 - 2x_2z + z^2) + \cdots + (x_n^2 - 2x_nz + z^2);$$

$$S(z) = (x_1^2 + x_2^2 + \cdots + x_n^2) - 2z(x_1 + x_2 + \cdots + x_n) + nz^2. \tag{3}$$

From (1), $x_1 + x_2 + \cdots + x_n = nA$. Hence, from (3),

$$S(z) = nz^2 - 2nAz + (x_1^2 + x_2^2 + \cdots + x_n^2). \tag{4}$$

By use of Section 229, it follows from (4) that the least value of $S(z)$ results if and only if $z = \dfrac{nA}{n} = A$. This proves Theorem I.

DEFINITION I. *The* **standard deviation,** *or* **root-mean-square deviation,** *of* $(x_1, x_2, \cdots, x_n)$ *is the square root of the arithmetic mean of the squares of the deviations of the x's from their arithmetic mean.*

If we let σ represent the standard deviation, then

$$\sigma^2 = \frac{(x_1 - A)^2 + (x_2 - A)^2 + \cdots + (x_n - A)^2}{n}. \tag{5}$$

From (2) and (5), $\sigma^2 = S(A)/n$. Hence, from (4) with $z = A$, we obtain the following useful formula:

$$\sigma^2 = \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n} - A^2. \tag{6}$$

From (5) we see that σ is *small* when the deviations of the x's from their arithmetic mean are *small.* Thus, if the x's cluster *closely* about their arithmetic mean, then their σ will be relatively *small,* while, if the x's are *widely dispersed,* their σ will be relatively

large. Hence, the standard deviation of a set of numbers may be used as a measure of their dispersion.

Illustration 2. The numbers (7, 6.5, 4, 5.2, 8, 7.5, 4.8, 5.8) are represented graphically below. By use of (1) and (5) we find $A = 6.1$ and $\sigma = 1.3$. Notice that the given numbers are scattered widely about their arithmetic mean, and that σ is relatively large.

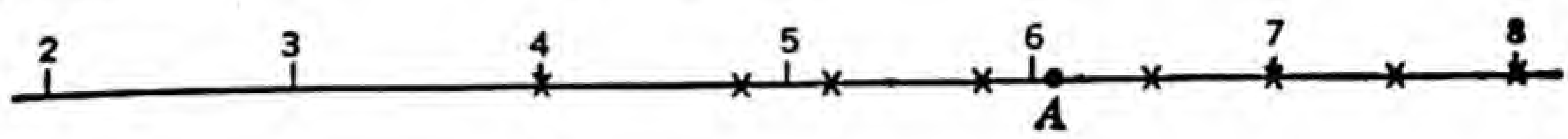


Illustration 3. The numbers (7.3, 7.6, 6.9, 6.8, 7.1, 7.5) are represented graphically below. For these numbers, $A = 7.2$ and $\sigma = .3$. Notice that the given numbers cluster closely about their arithmetic mean and that σ is relatively small.

231. Best approximation in the sense of least squares.

Illustration 1. Suppose that the following results, in degrees, have been obtained in six measurements of the latitude of a certain point on the earth: (73.9, 74.0, 74.0, 73.8, 74.1, 74.2). We may then ask "*on the basis of these measurements, what is our best estimate of the latitude?*" Before this is answered, we should *define* what we mean by a *best estimate*, because these words by themselves are indefinite.

Definition I. *A number w will be called the best approximation to a set of numbers $(x_1, x_2, \cdots, x_n)$, or the best representative for the set, in case the sum of the squares of the deviations of the x's from w is less than the sum resulting if w is replaced by any other number.*

In view of the possibility of other definitions of a *best approximation*, we shall say that Definition I describes the best approximation **in the sense of least squares.** In Definition I, the use of the *squares* of the deviations instead of the *deviations themselves* has the desirable effect of making the *sign* of a deviation immaterial. The numerical value of a deviation, and not its sign, is of importance in the sense of Definition I. On comparing Theorem I, Section 230, and Definition I, we see that, *in the sense of least squares, the best approximation to $(x_1, x_2, \cdots, x_n)$ is the arithmetic mean of the x's.*

Illustration 2. Our best approximation in Illustration 1 is

$$A = \tfrac{1}{6}(73.9 + 74.0 + 74.0 + 73.8 + 74.1 + 74.2) = 74.0$$

232. A notation for sums. To abbreviate a sum of a sequence of terms like $u_1 + u_2 + \cdots + u_n$ we write $\Sigma_{i=1}^{n} u_i$, which is read "*the sum of the numbers u_i for all values of i from* 1 *to n.*" We refer to capital sigma, Σ, as the **sign of summation,** and call i the *variable,* or the *index, of summation.* "Σ" abbreviates the word *sum.* Sometimes we omit writing the index of summation beside Σ.

ILLUSTRATION 1.
$$\Sigma_{j=1}^{6} v_j = v_1 + v_2 + v_3 + v_4 + v_5 + v_6. \tag{1}$$
$$\Sigma_{i=1}^{6} v_i = v_1 + v_2 + v_3 + v_4 + v_5 + v_6. \tag{2}$$
$$\Sigma_{x=1}^{n} x^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2.$$
From Section 230,
$$A = \frac{\Sigma_{i=1}^{n} x_i}{n}; \qquad \sigma = \sqrt{\frac{\Sigma_{i=1}^{n}(x_i - A)^2}{n}}. \tag{3}$$

Comment. From (1) and (2) we notice that the *letter* used for the *index of summation* has no effect on the value of the sum.

ILLUSTRATION 2. If $(x_1, x_2, \cdots, x_n)$ represent a set of values of x, then
$$\Sigma x = x_1 + x_2 + \cdots + x_n; \qquad \Sigma x^2 = x_1^2 + x_2^2 + \cdots + x_n^2. \tag{4}$$
In (4), we read "Σx" as the *sum of all values of x;* "Σx^2" means the *sum of the squares of all values of x.*

ILLUSTRATION 3. From formula 6, page 271, the standard deviation, σ, of $(x_1, x_2, \cdots, x_n)$ satisfies the equation
$$\sigma^2 = \frac{\Sigma x^2}{n} - A^2.$$

233. Use of a new origin for computing A and σ. Let c be selected arbitrarily and let $(y_1, y_2, \cdots, y_n)$ be the corresponding deviations of $(x_1, x_2, \cdots, x_n)$ from c; that is, $y_i = x_i - c$ in all cases. Since $x_i = y_i + c$, the arithmetic mean of the x's is given by
$$A = \frac{\Sigma x}{n} = \frac{(y_1 + c) + (y_2 + c) + \cdots + (y_n + c)}{n};$$
$$A = \frac{\Sigma y + nc}{n} = \frac{\Sigma y}{n} + c. \tag{1}$$

Let a represent the arithmetic mean of the y's; then we have $a = \dfrac{\Sigma y}{n}$ and, from (1),
$$A = a + c. \tag{2}$$
That is, if c is any constant, *the arithmetic mean of the x's equals* c *plus the arithmetic mean of the deviations of the x's from* c. Thus,

the relation $x_i = y_i + c$ leads to the similar relation $A = a + c$ between the arithmetic means of the x's and of the y's. To obtain a corresponding new formula for σ, we observe that

$$\begin{aligned}\Sigma x^2 &= (y_1 + c)^2 + (y_2 + c)^2 + \cdots + (y_n + c)^2 \\ &= (y_1^2 + y_2^2 + \cdots + y_n^2) + 2c(y_1 + y_2 + \cdots + y_n) + nc^2. \qquad (3)\end{aligned}$$

Since $\Sigma y = na$, from (3) we obtain

$$\Sigma x^2 = \Sigma y^2 + 2cna + nc^2. \qquad (4)$$

Since $\sigma^2 = \dfrac{\Sigma x^2}{n} - A^2$, from (2) and (4) we obtain

$$\sigma^2 = \frac{\Sigma y^2 + 2acn + nc^2}{n} - (a^2 + 2ac + c^2);\ \text{or}$$

$$\sigma^2 = \frac{\Sigma y^2}{n} - a^2. \qquad (5)$$

But, the right member of (5) is exactly the square of the standard deviation of $(y_1, y_2, \cdots, y_n)$ as given by (6) on page 271. Hence, if c is any constant, *the standard deviation of the x's is the same as the standard deviation of the deviations $(y_1, y_2, \cdots, y_n)$ of the x's from c.*

Note 1. A constant c, as introduced in this section, is sometimes referred to as the x-coordinate of an **arbitrary origin** for describing the data, because if two variables y and x are related by $y = x - c$, then $y = 0$ if $x = c$. Also, c is sometimes spoken of as an **approximate mean,** or an **assumed mean.** Frequently, the subtraction of a suitable number c from all x's of our data, or the use of a convenient new origin, greatly reduces the size of the numbers involved in computing A and σ.

ILLUSTRATION 1. To compute A and σ for the set of values of x in the adjoining table, we choose $c = 1900$ (which is a convenient number near to those of the given set). In the table, y represents the deviation of x from c. We have $a = \dfrac{\Sigma y}{7} = 1.76$. Hence, by use of (2), $A = 1901.76$. By use of (5),

x	y	y^2
1903.0	3.0	9.00
1903.0	3.0	9.00
1902.5	2.5	6.25
1900.6	.6	.36
1898.7	− 1.3	1.69
1905.2	5.2	27.04
1899.3	− .7	.49
	$\Sigma y = 12.3$	$\Sigma y^2 = 53.83$

$$\sigma^2 = \frac{53.83}{7} - (1.76)^2 = 4.59. \qquad \sigma = \sqrt{4.59} = 2.14.$$

EXERCISE 100

Find the minimum value of the function. Then verify the result by graphing the function.

1. $3x^2 - 12x + 4$. **2.** $5x^2 + 10x - 8$. **3.** $3x^2 - 9x + 1$.

Find the arithmetic mean of the numbers.

4. 175; 215; 310; 42; -50. **5.** -36; -28; 13; 27; 15.

Write the sum abbreviated by the symbol.

6. $\sum_{n=1}^{4} n^2$. **8.** $\sum_{n=1}^{6} a_n$. **10.** $\sum_{i=1}^{n} x_i^2$. **12.** $\sum_{i=1}^{n} x_i y_i$. **14.** $\sum_{i=1}^{6} (z - x_i)^2$.

7. $\sum_{n=1}^{5} n^3$. **9.** $\sum_{i=1}^{4} x_i$. **11.** $\sum_{j=1}^{n} b_j c_j$. **13.** $\sum_{n=1}^{7} \frac{1}{2} n^2$. **15.** $\sum_{j=2}^{5} (3j - 1)$.

16. If $(x_1, x_2, \cdots, x_n)$ are given values of a variable x, what sum is meant by Σx^3; $\Sigma 3x$; $\Sigma(x - c)$, where c is a constant?

17. If $(x_1, x_2, \cdots, x_n)$ and $(y_1, y_2, \cdots, y_n)$ are given values of two variables x and y, write the sum abbreviated by Σxy; Σx^2y^2; $\Sigma(x - y)$.

Abbreviate by use of Σ, using an index of summation

18. $n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2 + n_6^2 + n_7^2$.

19. $v_1^3 + v_2^3 + v_3^3 + v_4^3 + v_5^3 + v_6^3$.

20. $f_1x_1 + f_2x_2 + \cdots + f_nx_n$.

21. $3x_1^2 + 3x_2^2 + 3x_3^2 + \cdots + 3x_n^2$.

22. $(y_1 - a)^2 + (y_2 - a)^2 + \cdots + (y_n - a)^2$.

Find the standard deviation of the numbers by use of (5), *page* 274. *Then, plot the numbers and their arithmetic mean on a scale.*

23. -3, 3, 5, 9, 6. **25.** 3.5, 3.8, 4.1, 3.6, 3.9, 3.3, 4.5, 4.2, 3.7, 3.4.

24. 6, -3, 4, 9, 14. **26.** 2.8, 3.2, 3.3, 2.7, 2.6, 2.8, 3.4, 2.3, 2.9, 3.0.

Compute the standard deviation and the arithmetic mean of the numbers by use of a new origin for the data.

27. 1327, 1329, 1322, 1326, 1332, 1321, 1325, 1319.

28. 4267, 4264, 4271, 4273, 4266, 4265, 4268, 4269.

29. 3.17, 3.19, 3.16, 3.18, 3.20, 3.15, 3.17, 3.21, 3.14.

30. -4.28, -4.25, -4.29, -4.23, -4.22, -4.21, -4.24.

31. Suppose that a set of numbers consists of x_1 repeated f_1 times, x_2 repeated f_2 times, $\cdots$, x_n repeated f_n times (the number f_i is called the *frequency* of the number x_i). Show that

$$A = \frac{\Sigma fx}{\Sigma f}; \qquad \sigma^2 = \frac{\Sigma fx^2}{\Sigma f} - A^2.$$

32. The following table gives the monthly index number of the United States Department of Labor for retail food prices from July, 1942, to June, 1943. The index number is a percentage, based on average prices in 1935–1939 as 100%. Find A and σ for the given entries.

July, '42	Aug.	Sept.	Oct.	Nov.	Dec.	Jan., '43	Feb.	Mar.	Apr.	May	June
125	126	127	130	131	133	133	134	137	141	143	142

33. State and prove a theorem like that of Section 229, page 270, referring to the *largest* value of a quadratic function.

34. Prove that the sum of the deviations of $(x_1, x_2, \cdots, x_n)$ from their arithmetic mean is zero.

234. Approximate solution of a system of more than n linear equations in n unknowns. For illustration, consider the following system where x and y are the unknowns.

$$\text{I.}\quad \begin{cases} a_1x + b_1y = c_1, \\ a_2x + b_2y = c_2, \\ a_3x + b_3y = c_3. \end{cases}$$

Note 1. If the coefficients in (I) are selected at random, usually any two of the equations would have a single solution (x, y), but, it would be unusual if this pair of values satisfied the remaining equation. That is, in general, a system of more than two linear equations in two unknowns is inconsistent, as has been seen in Section 208, page 240.

Let h_1, h_2, and h_3 be defined as follows:

$$h_1 = a_1x + b_1y - c_1; \quad h_2 = a_2x + b_2y - c_2; \quad h_3 = a_3x + b_3y - c_3. \tag{1}$$

If (x, y) is a solution of (I), then $h_1 = h_2 = h_3 = 0$. Otherwise, the numerical value of any one of the h's indicates the extent to which (x, y) fails to satisfy a corresponding equation in (I). If (x, y) is a given pair of values, we call the h's the corresponding *residuals* for (I). Admitting that (I) is probably inconsistent, we propose the following problem:

Problem I. *To find a set of values of the unknowns such that the sum of the squares of the corresponding residuals is a minimum.*

Solution. 1. Let $f(x, y) = h_1^2 + h_2^2 + h_3^2$. From (1),

$$h_1^2 = a_1^2x^2 + b_1^2y^2 + c_1^2 - 2a_1c_1x - 2b_1c_1y + 2a_1b_1xy; \tag{2}$$

$$h_2^2 = a_2^2x^2 + b_2^2y^2 + c_2^2 - 2a_2c_2x - 2b_2c_2y + 2a_2b_2xy; \tag{3}$$

$$h_3^2 = a_3^2x^2 + b_3^2y^2 + c_3^2 - 2a_3c_3x - 2b_3c_3y + 2a_3b_3xy. \tag{4}$$

2. On adding (2), (3), and (4), we obtain

$$f(x, y) = x^2\Sigma a^2 + y^2\Sigma b^2 + \Sigma c^2 - 2x\Sigma ac - 2y\Sigma bc + 2xy\Sigma ab \qquad (5)$$

3. In $f(x, y)$, for the moment think of y as a constant. Then $f(x, y)$ is a quadratic function of x:

$$f(x, y) = x^2\Sigma a^2 + 2x(y\Sigma ab - \Sigma ac) + \textit{other terms}. \qquad (6)$$

4. If x has a value such that $f(x, y)$ has its minimum value, then, by Section 229, x must satisfy

$$x = -\frac{y\Sigma ab - \Sigma ac}{\Sigma a^2}; \qquad \textit{or} \qquad x\Sigma a^2 + y\Sigma ab = \Sigma ac. \qquad (7)$$

Similarly, by use of Section 229, we find that, if y has a value such that $f(x, y)$ has its minimum value, then y must satisfy

$$x\Sigma ab + y\Sigma b^2 = \Sigma bc. \qquad (8)$$

Hence, in order for (x, y) to be a solution of Problem I, it is necessary * that (x, y) satisfy the system

$$\text{II.} \quad \begin{cases} x\Sigma a^2 + y\Sigma ab = \Sigma ac, & (9) \\ x\Sigma ab + y\Sigma b^2 = \Sigma bc. & (10) \end{cases}$$

A solution (x, y) of (II) is an *approximate solution* of (I), and, for short, will be called *the solution of* (I) *in the sense of least squares.*

Note 2. The method by which we obtained (II) would establish the validity of the following method; we shall omit the demonstration.

METHOD I. *To obtain an approximate solution of a system of more than n linear equations in n unknowns x, y, ⋯ by the method of least squares.*

1. *Multiply each equation by the coefficient of x in it and add corresponding sides of the resulting equations; call the final equation thus obtained the* **normal equation** *corresponding to x.*

2. *Repeat Step* 1 *for each unknown in turn, thus obtaining as many normal equations as there are unknowns.*

3. *Solve the system consisting of the normal equations.*

ILLUSTRATION 1. For a system in three unknowns where each equation is of the form $a_ix + b_iy + c_iz = d_i$, the normal equations are

$$\begin{cases} x\Sigma a^2 + y\Sigma ab + z\Sigma ac = \Sigma ad, \\ x\Sigma ab + y\Sigma b^2 + z\Sigma bc = \Sigma bd, \\ x\Sigma ac + y\Sigma bc + z\Sigma c^2 = \Sigma cd. \end{cases}$$

* It is beyond the scope of this text to prove that [(9), (10)] are also *sufficient* conditions, but we shall assume this fact without proof.

ILLUSTRATION 2. In the following system in the unknowns a and b,

$$\begin{cases} 2a + b = 12, & (11)\\ a - b = 3, & (12)\\ 6a - 2b = 9, & (13)\end{cases}$$

to obtain the normal equation corresponding to a, we multiply both sides of (11) by 2, of (12) by 1, and of (13) by 6, and add:

$$\left.\begin{aligned} 4a + 2b &= 24,\\ a - b &= 3,\\ 36a - 12b &= 54.\end{aligned}\right\} \quad (add): \qquad 41a - 11b = 81. \qquad (14)$$

Similarly, we obtain the normal equation corresponding to b:

$$-11a + 6b = -9. \qquad (15)$$

In the sense of least squares, the best approximation to a solution of the system [(11), (12), (13)] is the solution of [(14), (15)], which we find to be

$$(a = 3.10; \quad b = 4.18).$$

Comment. With $(a = 3.10, b = 4.18)$ in [(11), (12), (13)], the *residuals* are found to be

$$h_1 = 2a + b - 12 = -1.62; \quad h_2 = -4.08; \quad h_3 = 1.24.$$

We obtain $h_1^2 + h_2^2 + h_3^2 = 20.81$. This is the *smallest* sum of the squares of the residuals which can result for any choice of values for (a, b).

EXERCISE 101

By the method of least squares, solve for the letters in the system.

1. $\begin{cases} 2a + 2b + 3 = 0,\\ -3a + b + 9 = 0,\\ a + 3b + 6 = 0.\end{cases}$

2. $\begin{cases} 2x + 3y - 2 = 0,\\ 2x - y + 3 = 0,\\ -x + 3y - 5 = 0.\end{cases}$

3. $\begin{cases} 2x + y = 4,\\ 3x - 2y = 0,\\ 5y = 7,\\ -x + y = -2.\end{cases}$

4. $\begin{cases} 3a = 5,\\ a - 2b = 3,\\ 3a + b = -2,\\ -a - 2b = 1.\end{cases}$

5. $\begin{cases} 2a - b + c = -1,\\ a + b - 2c = 3,\\ -3a + 2b = 4,\\ a - b + c = -2,\\ -2a - 3c = -1.\end{cases}$

6. $\begin{cases} x - 2y - z = 4,\\ 3y + 2z = -2,\\ -2x + y - z = -1,\\ 3x - z = 2,\\ -x - y + z = 1.\end{cases}$

7. By use of the symbol Σ, write the normal equations for the solution of the adjoining system for the unknowns a and b by the method of least squares ($x_1, x_2, \cdots, y_1, y_2, \cdots$ are given constants). Then, solve the normal equations for (a, b).

$$\begin{cases} y_1 = ax_1 + b,\\ y_2 = ax_2 + b,\\ y_3 = ax_3 + b,\\ y_4 = ax_4 + b,\\ y_5 = ax_5 + b.\end{cases}$$

235. Determination of polynomials with assigned values.

ILLUSTRATION 1. To find a and b so that the graph of $y = ax + b$ passes through $(1, -3)$ and $(-2, 3)$, we substitute the given values for x and y in $y = ax + b$:

$$\begin{aligned} &\text{when } (x = 1, y = -3): && -3 = a + b, \\ &\text{when } (x = -2, y = 3): && 3 = -2a + b. \end{aligned} \tag{1}$$

From (1), $(a = -2, b = -1)$; the desired line is

$$y = -2x - 1.$$

ILLUSTRATION 2. To determine a quadratic function $y = ax^2 + bx + c$ whose graph passes through $(1, -2)$, $(-1, 2)$, and $(2, 5)$, we would obtain three equations in the unknowns (a, b, c) by substituting $(x = 1, y = -2)$, etc. in $y = ax^2 + bx + c$. Then, we would solve for (a, b, c).

Note 1. Illustrations 1 and 2 are special cases of the following problem:

PROBLEM I. *To find a polynomial of degree n or less,*

$$y = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n, \tag{2}$$

whose graph passes through m given points.

SOLUTION. On substituting the coordinates of the given points in (2), we obtain m linear equations in the $(n + 1)$ unknown coefficients $a_0, a_1, \cdots, a_n$. Usually, this system of equations would have *one and only one solution if* $m = n + 1$, as in Illustrations 1 and 2; *infinitely many solutions if we have* $m < n + 1$; *no solution if* $m > n + 1$, because a system containing more equations than unknowns usually is inconsistent. In this case, we are led to consider approximate solutions, which are treated in the next section.

236. Curve fitting in the sense of least squares.

DEFINITION I. *A function $y = f(x)$ of a specified type will be called a best fitting function for a given set of points (x_1, y_1), (x_2, y_2), $\cdots$, (x_n, y_n), if the sum of the squares of the distances,* **measured parallel to the y-axis,** *from the graph of $f(x)$ to the points has the least value possible for this type of function.*

THE STRAIGHT LINE CASE OF DEFINITION I. Suppose that the specified type of function is $y = ax + b$, where a and b may be any constants. When $x = x_1$, the ordinate of the graph of the function is $ax_1 + b$. Let h_1 be the distance from the graph to (x_1, y_1): then, from Figure 32,

$$h_1 = y_1 - (ax_1 + b).$$

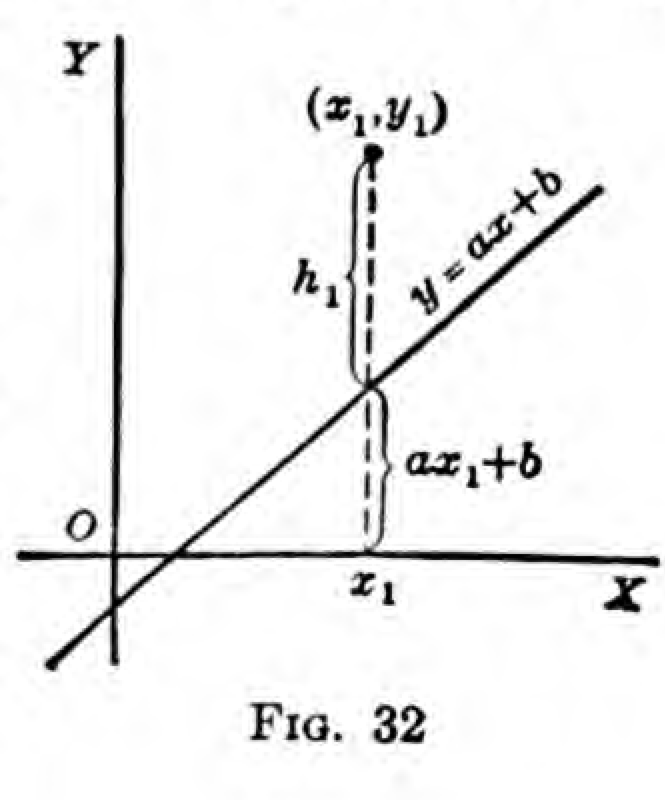


FIG. 32

If we think of $y = ax + b$ as a formula from

which we estimate the ordinates of the given points, then h_1 is the error, or residual, of our estimate when $x = x_1$. Thus, corresponding to the given points we have the errors

$$h_1 = y_1 - (ax_1 + b); \quad h_2 = y_2 - (ax_2 + b); \quad \cdots \quad ; \quad h_n = y_n - (ax_n + b).$$

By Definition I, $ax + b$ is a best fitting linear function if a and b have values for which Σh^2 has its least value. This condition on (a, b) is the same as that specified by the method of least squares, of Section 234, for the approximate solution of the following system in the variables (a, b):

$$\left.\begin{array}{l} y_1 = ax_1 + b, \\ y_2 = ax_2 + b, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ y_n = ax_n + b. \end{array}\right\} \tag{1}$$

Hence, to obtain the coefficients (a, b) of the best fitting function $y = ax + b$, we solve (1) for (a, b) by the method of least squares. We notice the useful fact that (1) can be obtained mechanically *by substituting the coordinates of each given point, in turn, for (x, y) in $y = ax + b$.*

Note 1. Since each coefficient of b in (1) is 1, the normal equations for (1) have the following simple form:

$$\Sigma xy = a\Sigma x^2 + b\Sigma x,$$
$$\Sigma y = a\Sigma x + nb.$$

EXAMPLE 1. Find the linear function $y = ax + b$ which best fits the points $(1, -2)$, $(-2, 3)$, $(4, -2)$, and $(-1, 1)$.

SOLUTION. On substituting in $y = ax + b$, we obtain

$$\begin{array}{ll} \text{when } (x = 1, y = -2): & -2 = a + b; \\ \text{when } (x = -2, y = 3): & 3 = -2a + b; \\ \text{when } (x = 4, y = -2): & -2 = 4a + b; \\ \text{when } (x = -1, y = 1): & 1 = -a + b. \end{array} \tag{2}$$

The normal equation for a: $\quad 22a + 2b = -17, \qquad (3)$

The normal equation for b: $\quad 2a + 4b = 0. \qquad (4)$

The solution of [(3), (4)] is $(a = -\frac{17}{21}, b = \frac{17}{42})$. The best fitting function is $y = -\frac{17}{21}x + \frac{17}{42}$. The student should graph this line and notice the extent to which the given points cluster about it.

Let $f(x)$ be a function involving a certain number of undetermined coefficients linearly. For instance, we might have

$$f(x) = a_0x^m + a_1x^{m-1} + \cdots + a_{m-1}x + a_m.$$

The following method may be justified by remarks like those for the straight line case of Definition I.

METHOD I. *To find the* best fitting function of the type $y = f(x)$ for a set of points $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$.*

1. *Substitute each pair (x_i, y_i) for (x, y) in $y = f(x)$ and thus obtain a system of linear equations in the coefficients of $f(x)$.*
2. *By the method of least squares, solve the system of Step 1 for the unknown coefficients, which give the best fitting function.*

EXAMPLE 2. Find the quadratic function $y = ax^2 + bx + c$ which best fits the points $(1, -1)$, $(-3, -1)$, $(-6, 2)$, and $(5, 3)$.

PARTIAL SOLUTION. We apply the preceding method. We obtain a system of equations by substituting in $y = ax^2 + bx + c$:

when $(x = 1, y = -1)$: $\quad -1 = a + b + c,$
when $(x = -3, y = -1)$: $\quad -1 = 9a - 3b + c,$
when $(x = -6, y = 2)$: $\quad 2 = 36a - 6b + c,$
when $(x = 5, y = 3)$: $\quad 3 = 25a + 5b + c.$

This system should be solved by the method of least squares.

237. Trend lines. We refer to a set of data as a **time series** if it gives the values of some variable y for a set of values of the *time t.*

ILLUSTRATION 1. The 2d row of the table gives the assets of all life insurance companies in the United States on December 31 of the various years. In the data, '12 means 1912. The unit for assets is \$100,000,000. Thus, 59 means \$5,900,000,000. This time series is represented by the black points in Figure 33.

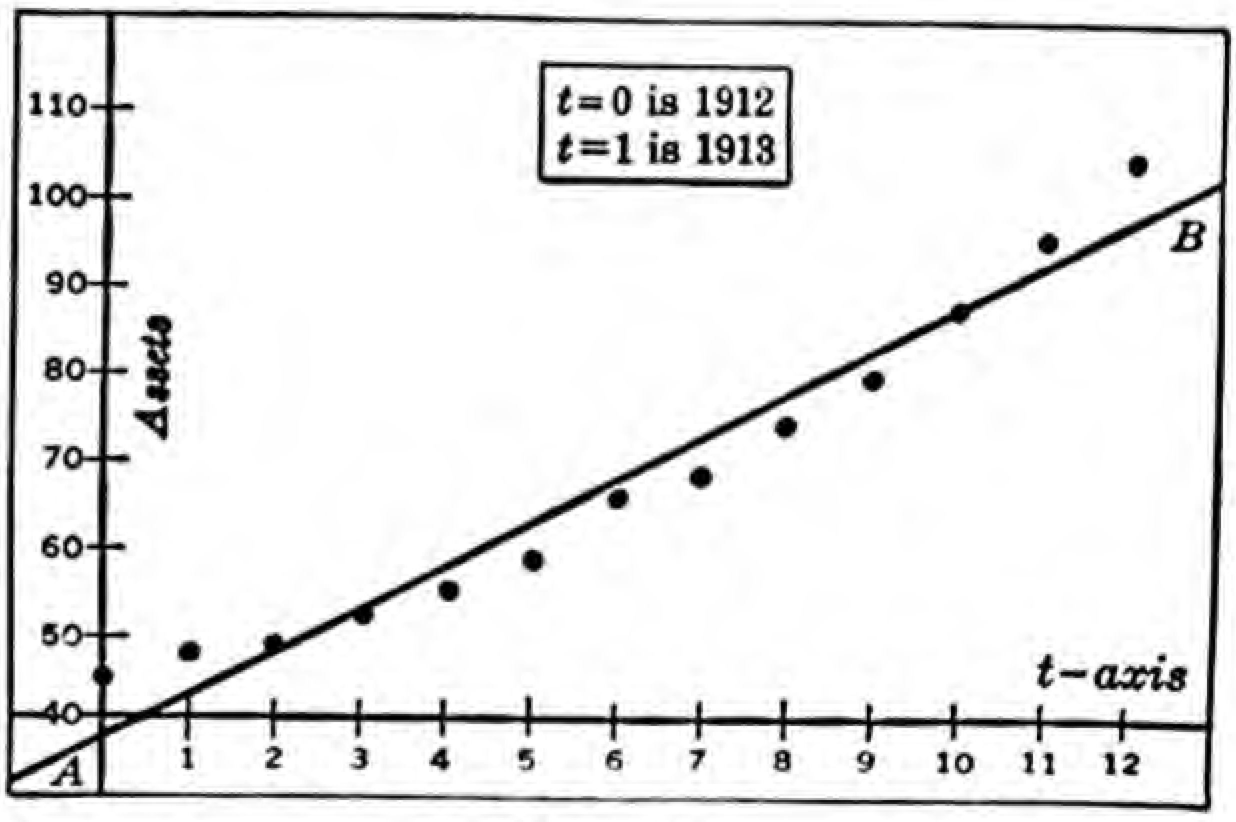


FIG. 33

'12	'13	'14	'15	'16	'17	'18	'19	'20	'21	'22	'23	'24
44	47	49	52	55	59	65	68	73	79	87	95	104

The graph of the best fitting linear function $y = at + b$ for a time series $(t_1, y_1), (t_2, y_2), \cdots$ is called the *trend line* for the series.

* The word "*a*" would be more proper than "*the*" because there might be *more than one*, or perhaps *no* best fitting function. However, in all cases met in this book a single result is obtainable.

EXAMPLE 1. Find the equation of the trend line in Illustration 1.

SOLUTION. 1. If we count time from 1912 as zero, the given points are $(t = 0, y = 44)$; $(t = 1, y = 47)$, $\cdots$.

2. To find the best fitting line of the form $y = at + b$, we write the following system:

$$\left.\begin{array}{lr} \text{when } (t = 0, y = 44): & 44 = b, \\ \text{when } (t = 1, y = 47): & 47 = a + b, \\ \text{when } (t = 2, y = 49): & 49 = 2a + b, \\ \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \text{when } (t = 12, y = 104): & 104 = 12a + b. \end{array}\right\} \qquad (1)$$

The solution of (1) by the method of least squares is $(a = 4.81, b = 38.6)$. The best fitting linear function is $y = 4.81t + 38.6$, whose graph is AB in Figure 33. The *trend line* is AB.

EXERCISE 102

Find a line of the form $y = ax + b$ whose graph passes through the points.

1. $(x = 2, y = -1)$; $(x = 3, y = 2)$. **2.** $(x = 3, y = 3)$; $(x = -3, y = 7)$.

Find a parabola of the form $y = ax^2 + bx + c$ whose graph passes through the given points (the x-axis is horizontal).

3. $(-1, 1)$; $(1, 3)$; $(3, -19)$. **4.** $(6, 2)$; $(-2, 10)$; $(4, -2)$.

Each column in a table gives the coordinates of a point. For each set of points, (1) find the best fitting function $y = ax + b$; (2) graph the function and the points on a coordinate system.

5.

x	5	-1	3	8
y	-1	4	2	1

7.

x	2	-3	0	5	5
y	-1	-2	1	2	4

6.

x	-2	1	2	4
y	2	1	2	5

8.

x	0	-2	3	3	1
y	-2	-2	3	1	1

Find the best fitting line of the form $y = mx$ for the data.

9. In Problem 7. **10.** In Problem 8.

11. The following table gives the daily average number of exchange messages, in hundreds of thousands, transmitted by the Bell Telephone System for the indicated years. Count time from 1925 as a zero, and determine the trend line. Then plot the data and the trend line, and estimate the number of messages for 1920 (the actual number was 31,800,000).*

* Due to the violent economic depression in the years following 1929, the trends observed in Examples 11 and 12 did not continue.

YEAR	1925	1926	1927	1928	1930
NUMBER OF MESSAGES	467	500	526	562	624

12. The table gives the total assets of savings and loan associations in the United States for certain years. The unit for assets is $100,000,000. Determine the trend line for the data. Plot the data and the trend line, and estimate the total assets for 1929.

YEAR	'20	'21	'22	'23	'24	'25	'26	'27	'28
ASSETS	25	29	33	39	48	55	63	72	80

Find the best fitting function of x of the form $y = ax^2 + bx + c$ for the points, and then graph the points and the parabola.

13. Points of Problem 6. **14.** Points of Problem 8.

15. The second row of the table gives the general wholesale price index number of the United States Department of Labor for the critical depression months from June, 1930, to June, 1931. Let 1 month be the unit of time, with June, 1930, as zero. Determine the trend line for the data. Graph the data and the trend line and estimate the value of the index number for July, 1931. A value like "86.8" means that the wholesale price level is 86.8% of the average level in 1926.

JUNE '30	JULY	AUG.	SEPT.	OCT.	NOV.	DEC.	JAN. '31	FEB.	MAR.	APR.	MAY	JUNE
86.8	84.0	84.0	84.2	82.6	80.4	78.4	77.0	75.5	74.5	73.3	71.3	70.0

16. Find the trend line for the data of Problem 32, Exercise 100; use 1 month as the unit of time, with July, 1942, as zero. Plot the trend line.

★*For each set of points, find the best fitting function* (1) *of the form $y = ax^2 + bx + c$, and* (2) *of the form $y = ax^3 + bx^2 + cx + d$; plot the resulting functions and the points.*

17.

x	0	− 1	2	− 2	3
y	− 5	− 3	− 3	− 7	13

18.

x	0	1	2	− 2	− 1
y	2	0	4	24	10

★*Find the best fitting function of the form $y = a \log x + b$ where a and b are undetermined constants and $\log x$ means $\log_{10} x$. Graph the resulting function and the data. Use four-place logarithms.*

19.

x	10	7	5	4
y	1.6	1.2	1.0	.6

20.

x	4	5	7	8	10
y	1	1.4	1.6	1.8	2.2

238. Regression lines. Think of the data (x_1, y_1), (x_2, y_2), $\cdots$, (x_n, y_n) plotted as points on an (x, y) coordinate plane. In the theory of statistics, the line of the form $y = ax + b$ which best fits the points in the sense of least squares, if distances are measured **parallel to the y-axis,** is called the *line of regression of y on x.* Impartial treatment of x and y leads us to determine a line of the form $x = cy + d$ which best fits the points if distances are measured **parallel to the x-axis;** this line is called the *line of regression of x on y.*

Note 1. Let σ_x be the standard deviation of the x's and σ_y that of the y's in the data. Then, in this and the next section, we assume that $\sigma_x \neq 0$ and $\sigma_y \neq 0$, which amounts to saying that *neither all the x's nor all the y's are equal.*

Line of regression of y on x, $y = ax + b$:	**Line of regression of x on y,** $x = cy + d$:
To determine a and b:	*To determine c and d:*
(1) $\begin{cases} y_1 = ax_1 + b, \\ y_2 = ax_2 + b, \\ \quad\cdots \\ y_n = ax_n + b. \end{cases}$	(3) $\begin{cases} x_1 = cy_1 + d, \\ x_2 = cy_2 + d, \\ \quad\cdots \\ x_n = cy_n + d. \end{cases}$
Normal equations for (a, b):	*Normal equations for* (c, d):
(2) $\begin{cases} \Sigma xy = a\Sigma x^2 + b\Sigma x, \\ \Sigma y = a\Sigma x + nb. \end{cases}$	(4) $\begin{cases} \Sigma xy = c\Sigma y^2 + d\Sigma y, \\ \Sigma x = c\Sigma y + nd. \end{cases}$

In an explicit problem, to find the lines of regression, we compute Σx, Σy, Σxy, Σx^2, and Σy^2 and use the results in (2) and (4) to find (a, b) and (c, d).

Note 2. In the following Example 1, it will be seen that the lines of regression for a given set of points may not coincide. Moreover, since a line of regression is a line of best fit, we obtain the following result: *if all of the given points lie on a straight line, then this line is the line of regression of y on x and also of x on y.*

EXAMPLE 1. Find the lines of regression for the following data:

x	-1	-1	1	1	1	2	3	4	5	4	6
y	3	5	1	4	6	2	4	1	3	-2	1

SOLUTION. 1. We find $\Sigma x = 25$; $\Sigma y = 28$; $\Sigma x^2 = 111$; $\Sigma y^2 = 122$; $\Sigma xy = 36$. From (2) and (4) of Note 1,

$$(5)\quad \begin{cases} 36 = 111a + 25b, \\ 28 = 25a + 11b; \end{cases}$$

$$(6)\quad \begin{cases} 36 = 122c + 28d, \\ 25 = 28c + 11d. \end{cases}$$

2. From (5), $(a = -.510,\ b = 3.70)$.

3. From (6), $(c = -.545,\ d = 3.66)$.

4. The line of regression of y on x is

$y = -.510x + 3.70$ (AB in Figure 34).

5. The line of regression of x on y is

$$x = -.545y + 3.66 \quad (CD \text{ in Figure } 34).$$

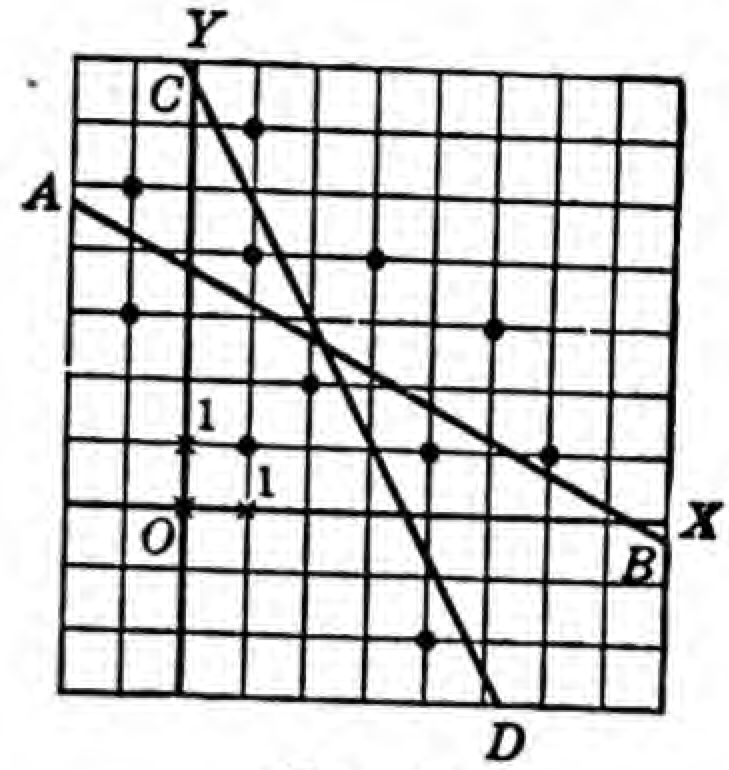


FIG. 34

Let us think of the given data (x_1, y_1), (x_2, y_2), $\cdots$ as merely a *sample* of the totality of pairs of corresponding values of x and y. Then, the line of regression of y on x, or $y = ax + b$, furnishes a formula *for estimating the value of* y *whenever* x *has an assigned value.* And, the line of regression of x on y, or $x = cy + d$, furnishes a formula *for estimating* x *if* y *is given.*

Let the arithmetic mean of $(x_1, x_2, \cdots, x_n)$ be $\bar{x}$, and of $(y_1, y_2, \cdots, y_n)$ be $\bar{y}$. Then, we shall call the point $(\bar{x}, \bar{y})$ the **average point** for the points (x_1, y_1), (x_2, y_2), $\cdots$, (x_n, y_n).

THEOREM I. *The regression lines pass through the average point.*

Proof. Since $\bar{x} = \Sigma x/n$ and $\bar{y} = \Sigma y/n$, on dividing by n in the second equation of (2) we obtain $\bar{y} = a\bar{x} + b$. This equality states that $y = ax + b$ is satisfied by $(x = \bar{x},\ y = \bar{y})$. A similar proof, employing the second equation of (4), shows that the line of regression of x on y also passes through $(\bar{x}, \bar{y})$.

ILLUSTRATION 1. In Example 1, $\bar{x} = 2.27$ and $\bar{y} = 2.55$. In Figure 34 we verify that the lines of regression pass through (2.27, 2.55).

Note 3. In addition to the lines of regression, we may consider a *third* line which best fits the given points in the sense of least squares if distances from the points to the line are measured *perpendicular to the line.* It can be proved * that this geometrically best fitting line intersects the lines of regression at the average point.

239. Coefficient of correlation.† Suppose that there exists a linear equation $gx + hy = k$, where g, h, and k are constants, which

* See B. H. CAMP's *Elementary Statistics*, D. C. HEATH AND COMPANY.

† For a complete treatment of the coefficient of correlation, see CAMP, *op. cit.*

is satisfied by all of the given points (x_1, y_1), (x_2, y_2), $\cdots$, (x_n, y_n). Then, it is said that the given data exhibit *perfect correlation* between x and y. That is, there is perfect correlation between x and y if and only if the given pairs of values of x and y, when plotted as points, fall on a straight line.

The symbol r, which is defined below, is in common use as a measure of the degree to which a given set of pairs of values of x and y approximates the condition of perfect correlation. In the following equation 1,

$$\bar{x} = \frac{\Sigma x}{n} \text{ and } \bar{y} = \frac{\Sigma y}{n}.$$

We call r *the coefficient of correlation of x and y.*

$$r = \frac{\Sigma_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\left[\Sigma_{i=1}^{n}(x_i - \bar{x})^2\right]\left[\Sigma_{i=1}^{n}(y_i - \bar{y})^2\right]}}. \tag{1}$$

If we let $X_i = x_i - \bar{x}$ and $Y_i = y_i - \bar{y}$, then

$$r = \frac{\Sigma XY}{\sqrt{\Sigma X^2 \Sigma Y^2}}. \tag{2}$$

ILLUSTRATION 1. In Example 1, Section 238, from (1) we find $r = -.53$.

Let the arithmetic mean of the squares of the distances from the given points to the line of regression of y on x be S^2, and to the line of regression of x on y be T^2, where these distances are measured parallel to the y-axis and to the x-axis, respectively. It can be proved* that

$$\begin{aligned} S^2 &= \sigma_y^2(1 - r^2); \\ T^2 &= \sigma_x^2(1 - r^2). \end{aligned} \tag{3}$$

The positive numbers S and T which satisfy (3) are the natural measures of the closeness with which the corresponding regression lines fit the given points.

THEOREM I. *For any set of points,* $-1 \leqq r \leqq 1$.

Proof. From (3) it follows that $(1 - r^2) \geqq 0$, because σ_y^2 is positive while $S^2 \geqq 0$. Hence, $1 \geqq r^2$; or $-1 \leqq r \leqq 1$.

THEOREM II. $r^2 = 1$ *when and only when the given points lie on a straight line. That is, there is perfect correlation when and only when* $r = \pm 1$.

* See CAMP, *op. cit.*, Chapter IX.

Proof. 1. From (3), if $r^2 = 1$ then $S^2 = 0$, and hence all of the given points lie on the line of regression of y on x.

2. Now, suppose that the points lie on a straight line. Then, this is the line of regression of y on x, and hence $S^2 = 0$. Thus,

$$\sigma_y^2(1 - r^2) = 0; \quad 1 - r^2 = 0; \quad r^2 = 1.$$

Note 1. Consider altering a given set (x_1, y_1), (x_2, y_2), $\cdots$ in such a way that σ_x and σ_y remain constant, but $|r|$ changes. Then, from (3), if $|r|$ *increases* the values of S and T *decrease*, or the fit of the regression lines to the data is *improved*. The *smaller* the numerical value of r, the *poorer* the fit. These remarks, in connection with Theorem II, show that r may appropriately be called a measure of the closeness with which either regression line fits the data. A more advanced discussion would lead to a similar conclusion concerning the connection of r with the closeness of fit of the geometrically best fitting line. We recognize that the extent to which the given points cluster about their geometrically best fitting line is the natural measure of the extent to which the data approximate the condition of perfect correlation. Hence, we are led to make the following statement:

The value of r for a set of pairs of values of x and y may be taken as a measure of the degree to which the data approximate the condition of perfect correlation. If $r = 1$ or $r = -1$, there is perfect correlation and the given pairs satisfy a linear equation in x and y. The smaller the numerical value of r, the less do the data approximate the ideal of perfect correlation.

Note 2. If $r = +1$, we say that there is perfect *positive* correlation between x and y. In this case, as a consequence of certain results which are obtained in problems of the next exercise, it follows that x and y satisfy an equation of the form $y = mx + b$, where $m > 0$. Hence, when $r = 1$, if (x_1, y_1) and (x_2, y_2) are pairs for which $x_1 > x_2$, then $y_1 > y_2$; that is, the *larger* the value of x, the *larger* is the corresponding value of y. If $r = -1$, we say that there is perfect *negative* correlation between x and y; in this case x and y satisfy an equation of the form $y = mx + b$ where $m < 0$. Hence, when $r = -1$, the *larger* the value of x, the *smaller* is the value of y. The *closer* r is to $+1$ (or, to -1), the *closer* do the data approximate the condition existing under perfect positive (or negative) correlation.

Note 3. We see that r is defined in terms of the deviations $x_i - \bar{x}$ and $y_i - \bar{y}$ of the x's and y's from their arithmetic means. If we add the same number, c, to all of the x's, we obtain new x's whose arithmetic mean is changed to $\bar{x} + c$, but the deviations of the new x's from their arithmetic mean are the same as the deviations of the original x's from $\bar{x}$. Hence, *the value of r is not altered if a constant is added to all of the x's, or to all of the y's.* This fact is frequently employed in computing r.

Note 4. Suppose that the x's and y's are the results of measuring two associated quantities. Then, it follows from (1) that *the value of r is independent of the unit in terms of which the x's, or the y's, were measured.* For, a change in the x-unit, or y-unit, would introduce a constant factor in the numerator and the same factor in the denominator for r, and hence would leave r unaltered. From this fact and the result of Note 3, it follows that r, for (x, y), is unaltered if (x, y) are transformed to (x', y') by equations $(x = hx' + k, y = ny' + v)$ where h, k, n, and v are constants.

EXERCISE 103

In each problem, find the lines of regression and the coefficient of correlation of x and y. Plot the data and the lines of regression.

1.

x	2	5	− 2	− 1
y	1	3	2	− 2

2.

x	1	4	− 1	− 2
y	3	4	3	6

3.

x	− 5	− 5	− 3	1	1	3	3	4	1
y	− 4	0	1	3	2	2	6	6	2

4.

x	− 2	− 1	− 1	0	1	3	4	3	2
y	3	2	5	4	1	1	− 1	1	2

5. The table below gives the average values of certain index numbers for the wholesale cost of *food* (x) and of *farm products* (y) in the United States for the years 1932 to 1943.* Compute the coefficient of correlation of x and y.

YEAR	'32	'33	'34	'35	'36	'37	'38	'39	'40	'41	'42	'43
x	61.0	60.5	70.5	83.7	82.1	85.5	73.6	70.4	71.3	82.7	99.6	106.6
y	48.2	51.4	65.3	78.8	80.9	86.4	68.5	65.3	67.7	82.4	105.9	122.6

Suppose that, on two examinations in a course, students I to X received the grades given in the table. In each problem, compute the coefficient of correlation of the grades on the two examinations.

6.

STUDENT	I	II	III	IV	V	VI	VII	VIII	IX	X
EXAM. 1	65	67	70	72	75	77	78	81	82	83
EXAM. 2	61	57	55	71	73	63	75	78	86	81

* From the *Statistical Abstract of the United States* for 1943. The indices are averages based upon the average for 1935–1939 as normal.

7.

STUDENT	I	II	III	IV	V	VI	VII	VIII	IX	X
EXAM. 1	83	82	81	78	77	75	72	70	67	65
EXAM. 2	63	55	81	71	86	75	61	78	73	57

Note 1. In Problem 6, r is near to $+1$, and we notice that this corresponds to the fact that, on the whole, a *high* grade on Examination 1 is associated with a *high* grade on Examination 2. In Problem 7, r is nearly zero, and this corresponds to the fact that there is little consistency between the grades on the two examinations. In Problems 6 and 7, where the data consist of so few pairs of values, a mere inspection of the data tells us at least as much as we learn from computing r. On the other hand, in a problem with a large amount of data, usually one can obtain little information by mere inspection. In such a case the coefficient of correlation becomes extremely useful.

8. By use of the definition of the standard deviation, prove that

$$r = \frac{\Sigma(x - \bar{x})(y - \bar{y})}{n\sigma_x\sigma_y}.$$

9. From (6), page 271, prove that $\sigma = \sqrt{n\Sigma x^2 - (\Sigma x)^2}/n$.

10. By use of Problems 8 and 9 show that

$$r = \frac{n\Sigma xy - (\Sigma x)(\Sigma y)}{\sqrt{n\Sigma x^2 - (\Sigma x)^2}\sqrt{n\Sigma y^2 - (\Sigma y)^2}}; \quad \text{or} \quad r = \frac{\Sigma xy - n\bar{x}\bar{y}}{n\sigma_x\sigma_y}.$$

11. By use of (2) and (4), page 284, show that

$$a = \frac{r\sigma_y}{\sigma_x}; \qquad c = \frac{r\sigma_x}{\sigma_y}; \qquad b = \frac{\bar{y}\sigma_x - r\bar{x}\sigma_y}{\sigma_x}.$$

240. The fitting of exponential and power functions* to a set of points (x_1, y_1), (x_2, y_2), $\cdots$, (x_n, y_n).

Note 1. Certain difficulties prevent us from easily finding the best fitting function (in the sense of least squares) of the form $y = ab^x$, or of the form $y = ax^b$. However, if the points are situated so that it is sensible to fit them with a type $y = ab^x$, or with $y = ax^b$, a satisfactory result can be found by use of the following methods.

If $y = ab^x$, and if we let $\log y = Y$, $\log a = A$, and $\log b = B$, then

$$\log y = \log a + x \log b, \quad \text{or} \quad Y = A + Bx.$$

Recall that the $(x, \log y)$-graph of $y = ab^x$ is the straight line which

* Sections 175 to 178, inclusive, are prerequisite here.

is the graph of $Y = A + Bx$ on an (x, Y) system of rectangular coordinates. This suggests the following procedure.

METHOD I. *To find numbers a and b so that the function $y = ab^x$ will approximately fit a set of given points.*

1. *Find the line $Y = A + Bx$ which best fits the corresponding points $(x, Y = \log y)$ in the sense of least squares.*

2. *By use of $A = \log a$ and $B = \log b$, find the values of a and b corresponding to (A, B) as found in Step 1.*

EXAMPLE 1. Determine a function $y = ab^x$ to fit the points $(x = 0, y = 210)$; $(x = 5, y = 130)$; $(x = 10, y = 70)$; $(x = 20, y = 30)$.

SOLUTION. 1. If $y = ab^x$, then

$$\log y = \log a + x \log b. \tag{1}$$

2. Let $\log a = A$ and $\log b = B$, and substitute the given values of (x, y) in (1):

$$\left.\begin{aligned} \log 210 &= A, \\ \log 130 &= A + 5B \\ \log\ \ 70 &= A + 10B, \\ \log\ \ 30 &= A + 20B. \end{aligned}\right\} \quad \text{Or} \quad \left.\begin{aligned} 2.3222 &= A, \\ 2.1139 &= A + 5B \\ 1.8451 &= A + 10B, \\ 1.4771 &= A + 20B. \end{aligned}\right\} \tag{2}$$

3. The least-square solution of (2) is $(A = 2.313, B = -.0426)$.

4. Since $\log a = 2.313$, hence $a = 206$.

5. Since $\log b = -.0426 = 9.9574 - 10$, hence $b = .907$.

6. The function obtained is $y = 206(.907^x)$.

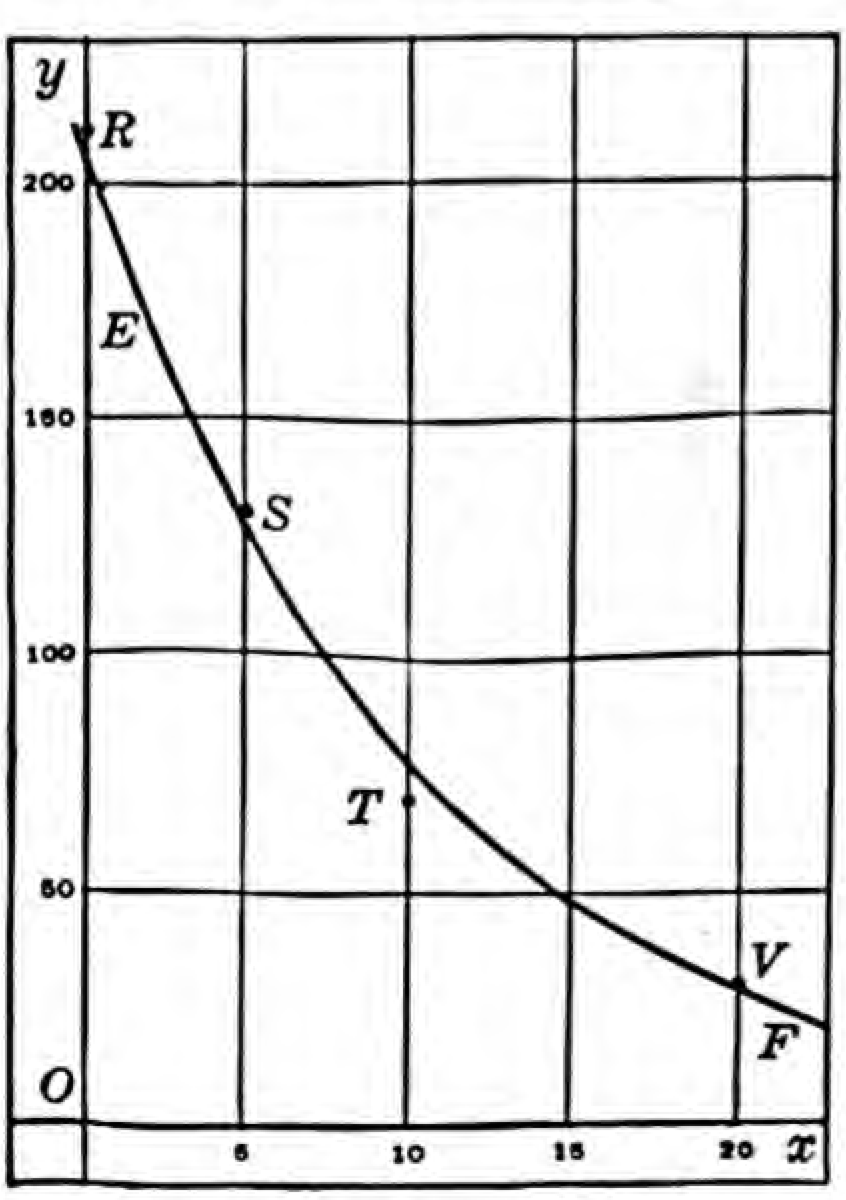


FIG. 35

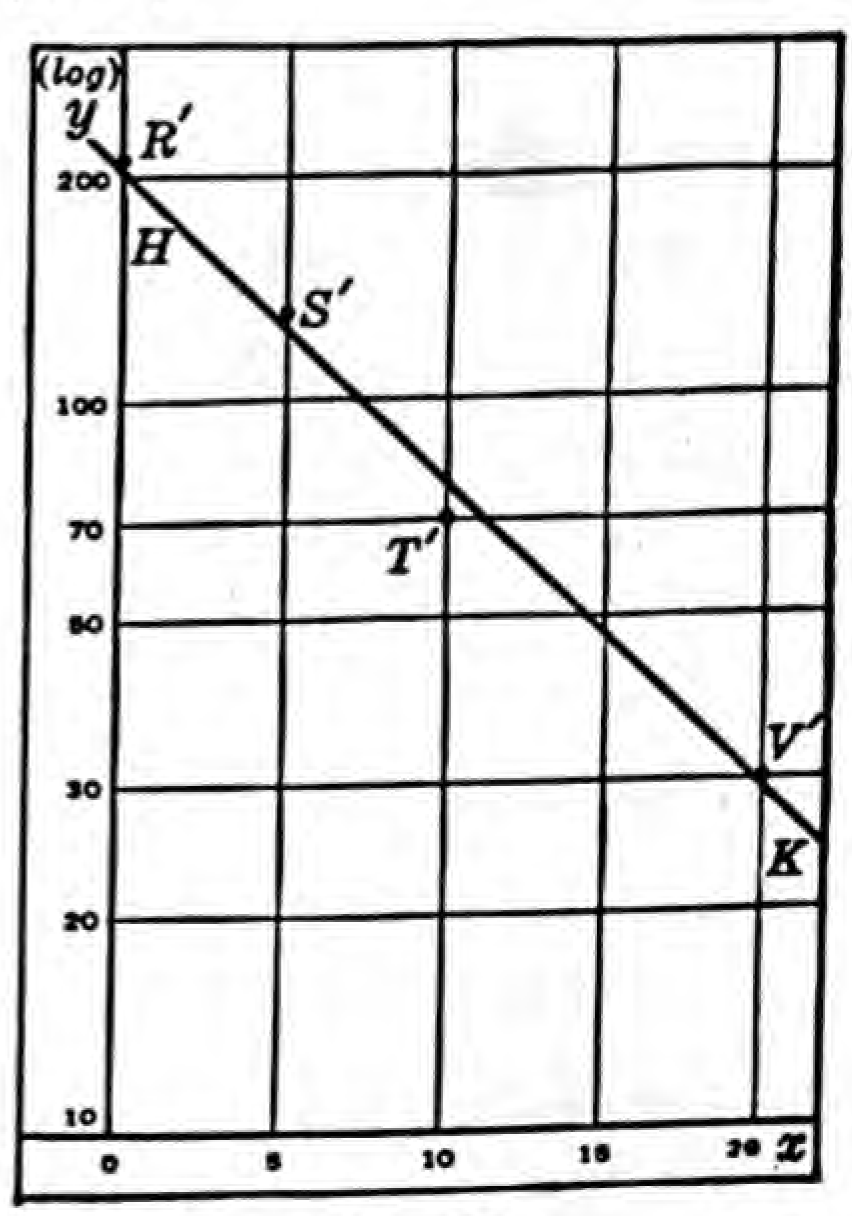


FIG. 36

Comment. In Figure 35, R, S, T, and V are the points given in Example 1, and EF is the graph of $y = 206(.907^x)$. In Figure 36, R', S', T', and V' are the points obtained on plotting the given pairs of values of x and y on an $(x, \log y)$ system of semilogarithmic coordinates. In Figure 36, line HK is the $(x, \log y)$-graph of $y = 206(.907^x)$, or is the ordinary graph of $Y = A + Bx$, where $(A = 2.313,\ B = -.0426)$. We do not claim that EF, in Figure 35, is a *best fitting curve, in the sense of least squares.* We merely say that the solution is useful because of the close geometrical fit.

Note 2. A method like that of this section would apply in fitting a function of the form $y = ab^xc^{x^2}$, or $y = ab^xc^{x^2}d^{x^3}$, etc.

If $y = ax^b$, then $\log y = \log a + b \log x$. If we let $\log y = Y$, $\log a = A$, and $\log x = X$, then $Y = A + bX$. Recall from page 191 that the $(\log x, \log y)$-graph of $y = ax^b$ is the straight line obtained as the graph of $Y = A + bX$ on an (X, Y) system of rectangular coordinates. This suggests the following procedure.

METHOD II. *To find values of a and b so that $y = ax^b$ will approximately fit a set of given points.*

1. *Find the coefficients A and b in the equation $Y = A + bX$ which best fits the points $(X = \log x,\ Y = \log y)$ in the sense of least squares.*
2. *Find a from $\log a = A$, where A has the value found in Step 1.*

EXAMPLE 2. Find a power function of the form $y = ax^b$ to fit the points $(x = 1, y = 3)$; $(x = 5, y = 350)$; $(x = 3, y = 100)$.

SOLUTION. 1. If $y = ax^b$, then
$$\log y = \log a + b \log x. \tag{3}$$

2. Let $\log a = A$ and substitute the given values of (x, y) in (3):
$$\left.\begin{aligned} \log 3 &= A + b \log 1, \\ \log 350 &= A + b \log 5, \\ \log 100 &= A + b \log 3. \end{aligned}\right\} \quad \text{Or} \quad \left.\begin{aligned} .4771 &= A, \\ 2.5441 &= A + .6990b, \\ 2 &= A + .4771b. \end{aligned}\right\} \tag{4}$$

The least-square solution of (4) is $(A = .500,\ b = 2.995)$.

3. Since $\log a = .500$, $a = 3.16$. The function obtained is $y = 3.16x^{2.995}$.

Note 3. Each of the equations $y = ab^x$ and $y = ax^b$ involves two undetermined constants. Hence, if we seek an equation of one of these types to fit *two* given points (x_1, y_1) and (x_2, y_2), either Method 1 or Method 2 leads us to *two* linear equations in two unknowns. Thus, if y_1 and y_2 are *positive* (so that we may use their logarithms), usually there is one and only one exponential function $y = ab^x$ whose graph passes through (x_1, y_1) and (x_2, y_2). If x_1, x_2, y_1, and y_2 are positive, usually there is one and only one power function $y = ax^b$ whose graph passes through (x_1, y_1) and (x_2, y_2).

Note 4. In curve fitting, to decide whether or not the exponential type would give a good fit, plot the given pairs (x, y) on a system of semilogarithmic coordinates. To decide whether or not a power function is the proper type to use, plot the pairs (x, y) on a $(\log x, \log y)$ system. If the resulting points cluster about some straight line, then a good fit probably will result by use of the specified type.

EXERCISE 104

In any problem, the first number of each pair within parentheses is a value of x and the second is a value of y. Find a function $y = ab^x$ to fit the points. Plot the resulting function and the given data (1) *on an (x, y) system of rectangular coordinates, and* (2) *on an $(x, \log y)$ system of semilogarithmic coordinates.*

1. (2, 12); (5, 96).

2. (− 2, 28); (− 3, 56).

3. (2, 20); (3, 50); (5, 500).

4. (2, 1.2); (4, 22); (6, 675).

5. (1, 240); (2, 110); (8, .35).

6. (1, 25); (3, 5); (7, .033).

Find a function $y = ax^b$ to fit the points. Plot the resulting function and the given data on an (x, y) system of rectangular coordinates and also on a double logarithmic $(\log x, \log y)$, *system of coordinates.*

7. (4, 118.4); (9, 899.1).

8. (8, 4.6); (27, 6.9).

9. (2, .11); (10, 4); (25, 16).

10. (8, 5); (10, 3); (216, .65).

In Problems 11 *and* 12, *find a function $y = ab^t$ to fit the data, where t is the time in years and y is the tabulated quantity; plot the resulting function and the data on a $(t, \log y)$ system of semilogarithmic coordinates.*

11. The table below gives the number of students (y) enrolled in high schools in continental United States per 100 of the population aged 14–17 years, for the indicated years. From the final graph, estimate the value of y for the year 1925. Use 1910 as $t = 0$ and 10 years as 1 unit for t.

YEAR	1900	1910	1920	1930	1940
$y =$	11	15	32	51	73

12. The table below gives the gross revenue in millions of dollars in the United States Postal Service for the indicated years. Use 1900 as $t = 0$. From the final graph, estimate the gross revenue for 1927.*

YEAR	1900	1905	1910	1915	1920	1925
POSTAL REVENUE	102	153	224	287	437	600

* The trend shown by these data was interrupted by the economic depression starting in 1930.

Appendix

Note 1. The Irrationality of $\sqrt{2}$

If there exists a rational number which is a square root of 2, then there exist two positive integers m and n, such that

$$\sqrt{2} = \frac{m}{n}, \tag{1}$$

where $\frac{m}{n}$ is a fraction in lowest terms. In other words, if $\sqrt{2}$ is rational there exist two integers m and n, **without a common factor,** such that (1) is true. Let us show that this assumption leads to a contradiction.

1. Square both sides of (1):

$$2 = \frac{m^2}{n^2}; \quad \text{or}$$

$$2n^2 = m^2. \tag{2}$$

We see that 2 is a factor of the left member of $2n^2 = m^2$; hence 2 is a factor of the right member. Therefore 2 is a factor of m because otherwise 2 could not be a factor of m^2. That is, $m = 2k$, where k is some positive integer.

2. Place $m = 2k$ in (2):

$$2n^2 = (2k)^2 = 4k^2;$$

$$n^2 = 2k^2. \tag{3}$$

Consider $n^2 = 2k^2$; since 2 is a factor of the right member, hence 2 is a factor of n.

3. We have shown in Steps 1 and 2 that m and n have 2 as a factor. This contradicts our original assumption that m and n had no common factor. Hence, the assumed equation 1 has led us to a contradiction, and it follows that (1) itself must be false. Therefore no rational number exists which is a square root of 2, or, *$\sqrt{2}$ is an irrational number.*

Comment. We easily verify that $(1.4)^2 = 1.96$; $(1.41)^2 = 1.9881$; $(1.414)^2 = 1.999396$; $(1.4142)^2 = 1.99996164$; etc. On considering the sequence of numbers

$$1.4, \quad 1.41, \quad 1.414, \quad 1.4142, \quad 1.41421, \quad \cdots, \tag{4}$$

we see that the square of each number in (4) is less than 2 but that, on proceeding to the right in (4), the squares of the numbers approach 2 as a limit. Each number in (4) is a rational number, a decimal fraction; we refer to these numbers in (4) as the successive decimal approximations to $\sqrt{2}$.

Note 2. Extension of the Index Laws to Rational Exponents

A complete proof that the index laws hold for any rational exponents could be constructed by showing, in succession, that the laws hold if the exponents are (1) any positive rational numbers and (2) zero, or positive or negative rational numbers. Without giving a complete discussion, we shall indicate the nature of the methods involved by proving some of the necessary theorems. For convenience in details, we shall assume that the *base* is positive. In our proofs, we use the index laws for positive integral exponents and the definitions of Sections 46, 47, and 48.

THEOREM I. *If m, n, and p are positive integers,* $\left(a^{\frac{m}{n}}\right)^p = a^{\frac{mp}{n}}$.

Proof. $\left(a^{\frac{m}{n}}\right)^p = \left[\left(a^{\frac{1}{n}}\right)^m\right]^p = \left(a^{\frac{1}{n}}\right)^{mp};$ [(4), page 37; (II), page 6]

or $\left(a^{\frac{m}{n}}\right)^p = a^{\frac{mp}{n}}.$ [(4), page 37]

THEOREM II. *If m, n, p, and q are positive integers, then*

$$a^{\frac{m}{n}}a^{\frac{p}{q}} = a^{\frac{m}{n}+\frac{p}{q}} = a^{\frac{mq+np}{nq}}.$$

Proof. $\left(a^{\frac{m}{n}}a^{\frac{p}{q}}\right)^{nq} = \left(a^{\frac{m}{n}}\right)^{nq}\left(a^{\frac{p}{q}}\right)^{nq}$ [(IV), page 6]

$= a^{mq}a^{pn}.$ (Theorem I)

Hence, $\left(a^{\frac{m}{n}}a^{\frac{p}{q}}\right)^{nq} = a^{mq+pn}.$ [(I), page 6]

Therefore, by the definition of an nqth root,

$$a^{\frac{m}{n}}a^{\frac{p}{q}} = \left(a^{mq+pn}\right)^{\frac{1}{nq}} = a^{\frac{mq+pn}{nq}}.$$

THEOREM III. $\left(a^{\frac{m}{n}}\right)^{\frac{p}{q}} = a^{\frac{mp}{nq}}.$

Suggestion for proof. Compute $\left[\left(a^{\frac{m}{n}}\right)^{\frac{p}{q}}\right]^{nq}$.

In the remainder of this note we shall assume that the index laws have been completely established for all positive rational exponents.

THEOREM IV. *Law* I *of Section* 14 *holds if the exponents are any positive or negative rational numbers.*

Comment. We are assuming that Law I has been established if both exponents are positive. Hence, it remains to show that, if h and k are any positive rational numbers, then $a^{-h}a^{-k} = a^{-h-k}$, and $a^ha^{-k} = a^{h-k}$.

Incomplete proof. By the definition of a negative power,

$$a^{-h}a^{-k} = \frac{1}{a^h}\cdot\frac{1}{a^k} = \frac{1}{a^{h+k}}; \quad \text{or}$$

$$a^{-h}a^{-k} = a^{-(h+k)}.$$

Note 3. Extension of De Moivre's Theorem to Rational Exponents

We defined $a^{\frac{m}{n}}$ on page 36 in case a is real and a^m has a real nth root; also, we have defined $(-P)^{\frac{1}{2}}$ as $i\sqrt{P}$ if $P > 0$. Beyond these cases, *no meaning has been given previously to* $a^{\frac{m}{n}}$. Now, if $z = R(\cos\theta + i\sin\theta)$, as a new definition let $z^{\frac{1}{n}}$ be *an n-valued symbol* to represent *any one of the* nth *roots of* z, obtained from (6) on page 131 with $k = 0, 1, 2, \cdots, (n-1)$. Similarly, if m/n is a rational number in lowest terms, with $n > 0$ for convenience, let $z^{\frac{m}{n}}$ represent *any one of the* nth *roots of* z^m. By De Moivre's Theorem, as generalized in Problem 20, page 130, $z^m = R^m(\cos m\theta + i\sin m\theta)$ for *all* positive and negative integers m. Hence, from (6) on page 131, **one** value of $z^{\frac{m}{n}}$ is

$$z^{\frac{m}{n}} = R^{\frac{m}{n}}\left(\cos\frac{m\theta}{n} + i\sin\frac{m\theta}{n}\right); \tag{1}$$

the other values are obtained (page 131) on replacing $m\theta$ successively by $(m\theta + 360°)$, $(m\theta + 720°)$, $\cdots$, $[m\theta + (n-1)360°]$. Formula 1 states that *if k is any rational number*, **one value** *of* $[R(\cos\theta + i\sin\theta)]^k$ is $R^k(\cos k\theta + i\sin k\theta)$. This statement generalizes De Moivre's Theorem of page 128 to the case of rational exponents.

Note 4. Proof of the Theorem on Imaginary Roots Occurring in Pairs

We wish to prove that if $(a + bi)$ is a root of $f(x) = 0$, with real coefficients, then $(a - bi)$ also is a root.

Proof. 1. Let
$$D(x) = [x - (a + bi)][x - (a - bi)] = x^2 - 2ax + a^2 + b^2.$$

2. Divide $f(x)$ by $D(x)$ until the remainder is a linear function, $cx + d$, and let $Q(x)$ be the quotient. Then,

$$f(x) \equiv D(x)Q(x) + cx + d, \tag{1}$$

where c and d are real because $D(x)$ has real coefficients. Since $(a + bi)$ is a root of $f(x) = 0$, hence $f(a + bi) = 0$. Since $D(x)$ has the factor $[x - (a + bi)]$, hence $D(a + bi) = 0$. Therefore, if we place $x = (a + bi)$ in (1), we obtain $0 = 0 + c(a + bi) + d$, or

$$(ac + d) + (bc)i = 0. \tag{2}$$

3. By use of statement 1 on page 121, it follows from (2) that

$$ac + d = 0, \quad \text{and} \quad bc = 0.$$

4. By hypothesis, $(a + bi)$ is imaginary and therefore $b \neq 0$. Since $bc = 0$, hence $c = 0$. Since $c = 0$ and $(ac + d) = 0$, hence $d = 0$. Therefore, the remainder $cx + d$ in (1) is zero, and thus $D(x)$ is a factor of $f(x)$. Hence, $f(x)$ has the factor $[x - (a - bi)]$ because this is a factor of $D(x)$. Therefore, $f(x)$ is zero when $x = a - bi$, or $(a - bi)$ also is a root of $f(x) = 0$.

Note 5. Proof of the Auxiliary Theorem for Descartes' Rule of Signs

THEOREM. *If $g(x)$ is any polynomial and r is positive, then $(x - r)g(x)$ has at least one more variation of sign than $g(x)$.*

Proof. 1. Suppose that*

$$g(x) = + + + + - - - - + + + + - - -, \qquad (1)$$

where in place of each term we merely write its sign. Then,

$$\begin{array}{rcl} xg(x) &=& + + + + - - - - + + + + - - - ; \qquad (2) \\ -rg(x) &=& - - - - + + + + - - - - + + +. \qquad (3) \\ \hline (x - r)g(x) &=& + \pm \pm \pm - \pm \pm \pm + \pm \pm \pm - \pm \pm +. \qquad (4) \end{array}$$

To obtain (4), we add corresponding members of (2) and (3). In (2), (3), and (4), we arranged each right member in descending powers of x and then wrote merely the sign of each term in place of the term itself. The signs for terms involving the same power of x are arranged in columns in (2), (3), and (4).

2. The first of each group of "+" terms in (2) is part of a "+" term in (4). The first of each group of "−" terms in (2) is part of a "−" term in (4). The intermediate terms are indicated "±" in (4) because each one is +, or −, or zero, depending on the values of r and of certain coefficients in $g(x)$.

3. We shall not *overestimate* the number of variations of sign in $(x - r)g(x)$ if we assume, in (4), that each ambiguous sign is the same as the preceding unambiguous sign. Under this assumption, the signs of $(x - r)g(x)$ are

$$+ + + + - - - - + + + + - - - +. \qquad (5)$$

These are the signs of $g(x)$ with an additional sign at the right-hand end which is unlike the last sign in $g(x)$, because the last term in $(x - r)g(x)$ equals the last term in $g(x)$ multiplied by $-r$.

* The statements of this proof are general in their application, even though they refer to a special polynomial whose signs are given in (1). In reading the proof for the first time, assume that in $g(x)$ no power of x is missing. If certain powers of x are missing in $g(x)$, the proof holds without alteration if a zero is listed in (1), (2), and (3) for each missing power.

4. Hence, in (5) we have all the variations of sign of $g(x)$, with one additional variation due to the last sign in (5). Therefore, $(x - r)g(x)$ has *at least* one more variation of sign than $g(x)$.

Note 1. A more refined discussion would show that, if $(x - r)g(x)$ has more variations of sign than $g(x)$, then the excess is an even number of variations. On the basis of this fact, Descartes' Rule of Signs can be stated in the following strong form: *The number of positive roots of $f(x) = 0$ cannot exceed the number of variations of sign in $f(x)$ and, in any case, differs from the number of variations by an* **even integer.**

Note 6. Theorems on Inversions and Related Proofs

THEOREM I. *In any permutation of integers (or letters), if two adjacent integers (or letters) are interchanged, the number of inversions is increased by or decreased by* 1.

ILLUSTRATION 1. In (1, 5, 3, 2, 4) there are four inversions; on interchanging 2 and 3, we obtain (1, 5, 2, 3, 4), with only three inversions.

*Proof.** Let x and y be adjacent integers in the permutation $AxyB$, where A denotes the group of integers which precede and B denotes those that follow xy. On interchanging x and y we obtain $AyxB$. Any inversions in A, or in B, and any due to the fact that A precedes x and y, or that Axy precedes B, are present in both $AxyB$ and $AyxB$. The only change in inversions is due to this: if xy is an inversion, then yx is not, and $AxyB$ has one more inversion than $AyxB$; or, if xy is not an inversion, then yx is, and hence $AxyB$ has one less inversion than $AyxB$.

In dealing with determinants, we represent elements by letters with subscripts attached, as a_1, b_3, c_2, etc. Let T denote any product of such elements in which all letters and subscripts are different. In T, let s and l be the number of inversions in the subscripts and in the letters, respectively.

THEOREM II. *If two adjacent factors in T are interchanged, then the combined number $(s + l)$ of inversions in subscripts and in letters is increased by $+ 2$, or by $- 2$, or by* 0.

ILLUSTRATION 2. If $T = a_1b_4d_3c_2$, then $(s + l) = 3 + 1 = 4$; on interchanging d_3 and c_2, we obtain $T = a_1b_4c_2d_3$, where $(s + l) = 2 + 0 = 2$.

Proof. If we interchange two adjacent symbols in T, we interchange two adjacent letters and two adjacent subscripts. Hence, by Theorem I, we increase s by ± 1 and l by ± 1. Therefore, we increase $(s + l)$ by $(\pm 1 \pm 1)$, which is either $+ 2$, or $- 2$, or 0.

THEOREM III. *For any product T, the combined number of inversions in letters and in subscripts is either always even, or always odd, regardless of the order of the factors of T.*

* This proof obviously applies to the case of the theorem referring to letters.

Proof. Any order for the factors can be obtained from any other order by successive interchanges of adjacent factors. Each interchange increases $(s + l)$ by $+ 2$, or $- 2$, or 0, each of which is *even*. Hence, if $(s + l)$ is *even* (or *odd*) for one order of the factors, it is *even* (or *odd*) for all orders.

Proof of Property I, page 229. In a given determinant D, suppose that the letters used in the columns are in alphabetical order from left to right. Let D' be obtained by interchanging corresponding rows and columns of D. We wish to show that $D = D'$.

Write out the terms of D with their factors arranged as directed in Step 2 of Definition I, page 227, and the terms of D' with their factors arranged as directed in Note 1 on page 228. Then, except perhaps for its sign, any term T' of D' can be obtained by rearranging the factors of a corresponding term T of D.

Illustration 3. If $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, and $D' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$,

one term of D is $T = - a_2b_1c_3$ and the corresponding term of D' is

$$T' = - b_1a_2c_3.$$

There are no inversions among the letters in T, nor among the subscripts in T'. Moreover, by Theorem III, the combined number of inversions among subscripts and letters is either *even in both T and T'* or else *odd in both.* Hence, the number of inversions among *letters* in T' and the number of inversions among *subscripts* in T are either *both even* or *both odd.* Hence, T' has the same sign in the expansion of D' that T has in D. Therefore, $D = D'$, because terms of D and of D' with the same factors also have the same sign.

Proof of Property V, page 229. Part I. First, let us prove that *if D' is obtained by interchanging two* **adjacent** *columns of D, then $D = - D'$.* Write out the terms of D and of D' with their factors arranged as directed in Step 2 of Definition I, page 227. Apart perhaps from sign, any term T' of D' can be obtained from a corresponding term T of D by interchanging two *adjacent letters* in T. This also interchanges two *adjacent subscripts* in T.

Illustration 4. If $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, and $D' = \begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix}$,

one term of D is $T = a_3b_1c_2$; the corresponding term T' of D' is $T' = - a_3c_2b_1$.

Hence, by Theorem I, the number of inversions among the subscripts in T' is *even* or *odd* according as the number of inversions among the subscripts in T is *odd* or *even*. Therefore, by Step 2 in Definition I, corresponding terms in D and D' have opposite signs. Hence, it follows that $D = - D'$.

Part II. Now, suppose that U and V are any two columns of D, with U to the left of V and with h columns between U and V. We can accomplish the interchange of U and V by $(2h + 1)$ successive interchanges of adjacent columns as follows:

Interchange U, in succession, with each of the h columns between U and V. This brings U next to V. Then, interchange V, in succession, with U, and with the h columns originally between U and V.

By Part I, the interchange of U and V causes $(2h + 1)$ changes, or an *odd* number of changes in the sign of D. Hence, the interchange of U and V changes the sign of D.

Proof of Property IX, page 231. For simplicity in writing, we shall deal with the special case where D is of the 4th order. Let

$$D = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}; \qquad \text{then} \qquad A_1 = \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix}.$$

Any term of D containing a_1 is of the form $\pm a_1b_ic_jd_k$, where (i, j, k) is a permutation of (2, 3, 4). Hence, there are 6 different terms involving a_1, because there are 3 ! or 6 permutations of (2, 3, 4) taken all at a time.

Write out the terms of A_1, with factors arranged as directed by Step 2 of Definition I, and let T be any term. Then, apart perhaps from sign, a_1T is a term of D, for it has one and only one factor from each row and column of D.

Illustration 5. $-b_2c_4d_3$ is a term of A_1; $-a_1b_2c_4d_3$ is a term of D.

Since the subscripts in T are a permutation of (2, 3, 4), the number of inversions in the subscripts of T is the same as in a_1T, because the subscript 1 on a_1 does not change the number of inversions. Hence, the sign of T in A_1 is the correct sign for a_1T in the expansion of D. Therefore, since A_1 contains 3 ! or 6 different terms, the 6 terms of the product a_1A_1 are the 6 terms of D which involve a_1.

Tables

TABLE I. POWERS AND ROOTS

No.	Sq.	Sq. Root	Cube	Cube Root	No.	Sq.	Sq. Root	Cube	Cube Root
1	1	1.000	1	1.000	**51**	2,601	7.141	132,651	3.708
2	4	1.414	8	1.260	**52**	2,704	7.211	140,608	3.732
3	9	1.732	27	1.442	**53**	2,809	7.280	148,877	3.756
4	16	2.000	64	1.587	**54**	2,916	7.348	157,464	3.780
5	25	2.236	125	1.710	**55**	3,025	7.416	166,375	3.803
6	36	2.449	216	1.817	**56**	3,136	7.483	175,616	3.826
7	49	2.646	343	1.913	**57**	3,249	7.550	185,193	3.848
8	64	2.828	512	2.000	**58**	3,364	7.616	195,112	3.871
9	81	3.000	729	2.080	**59**	3,481	7.681	205,379	3.893
10	100	3.162	1,000	2.154	**60**	3,600	7.746	216,000	3.915
11	121	3.317	1,331	2.224	**61**	3,721	7.810	226,981	3.936
12	144	3.464	1,728	2.289	**62**	3,844	7.874	238,328	3.958
13	169	3.606	2,197	2.351	**63**	3,969	7.937	250,047	3.979
14	196	3.742	2,744	2.410	**64**	4,096	8.000	262,144	4.000
15	225	3.873	3,375	2.466	**65**	4,225	8.062	274,625	4.021
16	256	4.000	4,096	2.520	**66**	4,356	8.124	287,496	4.041
17	289	4.123	4,913	2.571	**67**	4,489	8.185	300,763	4.062
18	324	4.243	5,832	2.621	**68**	4,624	8.246	314,432	4.082
19	361	4.359	6,859	2.668	**69**	4,761	8.307	328,509	4.102
20	400	4.472	8,000	2.714	**70**	4,900	8.367	343,000	4.121
21	441	4.583	9,261	2.759	**71**	5,041	8.426	357,911	4.141
22	484	4.690	10,648	2.802	**72**	5,184	8.485	373,248	4.160
23	529	4.796	12,167	2.844	**73**	5,329	8.544	389,017	4.179
24	576	4.899	13,824	2.884	**74**	5,476	8.602	405,224	4.198
25	625	5.000	15,625	2.924	**75**	5,625	8.660	421,875	4.217
26	676	5.099	17,576	2.962	**76**	5,776	8.718	438,976	4.236
27	729	5.196	19,683	3.000	**77**	5,929	8.775	456,533	4.254
28	784	5.291	21,952	3.037	**78**	6,084	8.832	474,552	4.273
29	841	5.385	24,389	3.072	**79**	6,241	8.888	493,039	4.291
30	900	5.477	27,000	3.107	**80**	6,400	8.944	512,000	4.309
31	961	5.568	29,791	3.141	**81**	6,561	9.000	531,441	4.327
32	1,024	5.657	32,768	3.175	**82**	6,724	9.055	551,368	4.344
33	1,089	5.745	35,937	3.208	**83**	6,889	9.110	571,787	4.362
34	1,156	5.831	39,304	3.240	**84**	7,056	9.165	592,704	4.380
35	1,225	5.916	42,875	3.271	**85**	7,225	9.220	614,125	4.397
36	1,296	6.000	46,656	3.302	**86**	7,396	9.274	636,056	4.414
37	1,369	6.083	50,653	3.332	**87**	7,569	9.327	658,503	4.431
38	1,444	6.164	54,872	3.362	**88**	7,744	9.381	681,472	4.448
39	1,521	6.245	59,319	3.391	**89**	7,921	9.434	704,969	4.465
40	1,600	6.325	64,000	3.420	**90**	8,100	9.487	729,000	4.481
41	1,681	6.403	68,921	3.448	**91**	8,281	9.539	753,571	4.498
42	1,764	6.481	74,088	3.476	**92**	8,464	9.592	778,688	4.514
43	1,849	6.557	79,507	3.503	**93**	8,649	9.644	804,357	4.531
44	1,936	6.633	85,184	3.530	**94**	8,836	9.695	830,584	4.547
45	2,025	6.708	91,125	3.557	**95**	9,025	9.747	857,375	4.563
46	2,116	6.782	97,336	3.583	**96**	9,216	9.798	884,736	4.579
47	2,209	6.856	103,823	3.609	**97**	9,409	9.849	912,673	4.595
48	2,304	6.928	110,592	3.634	**98**	9,604	9.899	941,192	4.610
49	2,401	7.000	117,649	3.659	**99**	9,801	9.950	970,299	4.626
50	2,500	7.071	125,000	3.684	**100**	10,000	10.000	1,000,000	4.642

TABLE II. FOUR-PLACE LOGARITHMS OF NUMBERS

N	0	1	2	3	4	5	6	7	8	9
10	.0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	.0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	.0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	.1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	.1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	.1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	.2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	.2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	.2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	.2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	.3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
N	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	43
1	4.3
2	8.6
3	12.9
4	17.2
5	21.5
6	25.8
7	30.1
8	34.4
9	38.7

	42	41	40	39		38	37	36		35	34	33	32	
1	4.2	4.1	4.0	3.9	1	3.8	3.7	3.6	1	3.5	3.4	3.3	3.2	1
2	8.4	8.2	8.0	7.8	2	7.6	7.4	7.2	2	7.0	6.8	6.6	6.4	2
3	12.6	12.3	12.0	11.7	3	11.4	11.1	10.8	3	10.5	10.2	9.9	9.6	3
4	16.8	16.4	16.0	15.6	4	15.2	14.8	14.4	4	14.0	13.6	13.2	12.8	4
5	21.0	20.5	20.0	19.5	5	19.0	18.5	18.0	5	17.5	17.0	16.5	16.0	5
6	25.2	24.6	24.0	23.4	6	22.8	22.2	21.6	6	21.0	20.4	19.8	19.2	6
7	29.4	28.7	28.0	27.3	7	26.6	25.9	25.2	7	24.5	23.8	23.1	22.4	7
8	33.6	32.8	32.0	31.2	8	30.4	29.6	28.8	8	28.0	27.2	26.4	25.6	8
9	37.8	36.9	36.0	35.1	9	34.2	33.3	32.4	9	31.5	30.6	29.7	28.8	9

	31	30	29	28		27	26	25		24	23	22	21	
1	3.1	3.0	2.9	2.8	1	2.7	2.6	2.5	1	2.4	2.3	2.2	2.1	1
2	6.2	6.0	5.8	5.6	2	5.4	5.2	5.0	2	4.8	4.6	4.4	4.2	2
3	9.3	9.0	8.7	8.4	3	8.1	7.8	7.5	3	7.2	6.9	6.6	6.3	3
4	12.4	12.0	11.6	11.2	4	10.8	10.4	10.0	4	9.6	9.2	8.8	8.4	4
5	15.5	15.0	14.5	14.0	5	13.5	13.0	12.5	5	12.0	11.5	11.0	10.5	5
6	18.6	18.0	17.4	16.8	6	16.2	15.6	15.0	6	14.4	13.8	13.2	12.6	6
7	21.7	21.0	20.3	19.6	7	18.9	18.2	17.5	7	16.8	16.1	15.4	14.7	7
8	24.8	24.0	23.2	22.4	8	21.6	20.8	20.0	8	19.2	18.4	17.6	16.8	8
9	27.9	27.0	26.1	25.2	9	24.3	23.4	22.5	9	21.6	20.7	19.8	18.9	9

	20	19	18	17		16	15	14		13	12	11	10	
1	2.0	1.9	1.8	1.7	1	1.6	1.5	1.4	1	1.3	1.2	1.1	1.0	1
2	4.0	3.8	3.6	3.4	2	3.2	3.0	2.8	2	2.6	2.4	2.2	2.0	2
3	6.0	5.7	5.4	5.1	3	4.8	4.5	4.2	3	3.9	3.6	3.3	3.0	3
4	8.0	7.6	7.2	6.8	4	6.4	6.0	5.6	4	5.2	4.8	4.4	4.0	4
5	10.0	9.5	9.0	8.5	5	8.0	7.5	7.0	5	6.5	6.0	5.5	5.0	5
6	12.0	11.4	10.8	10.2	6	9.6	9.0	8.4	6	7.8	7.2	6.6	6.0	6
7	14.0	13.3	12.6	11.9	7	11.2	10.5	9.8	7	9.1	8.4	7.7	7.0	7
8	16.0	15.2	14.4	13.6	8	12.8	12.0	11.2	8	10.4	9.6	8.8	8.0	8
9	18.0	17.1	16.2	15.3	9	14.4	13.5	12.6	9	11.7	10.8	9.9	9.0	9

Proportional Parts

TABLE II. FOUR-PLACE LOGARITHMS OF NUMBERS

N	0	1	2	3	4	5	6	7	8	9
20	.3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	.3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	.3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	.3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	.3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	.3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	.4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	.4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	.4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	.4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	.4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	.4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	.5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	.5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	.5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	.5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	.5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	.5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	.5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	.5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	.6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	.6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	.6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	.6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	.6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	.6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	.6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	.6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	.6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	.6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	.6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	.7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	.7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	.7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	.7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
55	.7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	.7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	.7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	.7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	.7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	.7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
N	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	9
1	0.9
2	1.8
3	2.7
4	3.6
5	4.5
6	5.4
7	6.3
8	7.2
9	8.1

	8
1	0.8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6
8	6.4
9	7.2

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

Prop. Pts.

TABLE II. FOUR-PLACE LOGARITHMS OF NUMBERS

N	0	1	2	3	4	5	6	7	8	9
60	.7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	.7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	.7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	.7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	.8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	.8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	.8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	.8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	.8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	.8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	.8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	.8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	.8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	.8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	.8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	.8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	.8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	.8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	.8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	.8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	.9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	.9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	.9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	.9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	.9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	.9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	.9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	.9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	.9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	.9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	.9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	.9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	.9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	.9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	.9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	.9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	.9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	.9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	.9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	.9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
N	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	8
1	0.8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6
8	6.4
9	7.2

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

	4
1	0.4
2	0.8
3	1.2
4	1.6
5	2.0
6	2.4
7	2.8
8	3.2
9	3.6

Prop. Pts.

TABLE III. AMERICAN EXPERIENCE TABLE OF MORTALITY

Age	Number Living	Number Dying	Yearly Probability of Dying	Yearly Probability of Living	Age	Number Living	Number Dying	Yearly Probability of Dying	Yearly Probability of Living
10	100,000	749	0.007 490	0.992 510	53	66,797	1,091	0.016 333	0.983 667
11	99,251	746	0.007 516	0.992 484	54	65,706	1,143	0.017 396	0.982 604
12	98,505	743	0.007 543	0.992 457	55	64,563	1,199	0.018 571	0.981 429
13	97,762	740	0.007 569	0.992 431	56	63,364	1,260	0.019 885	0.980 115
14	97,022	737	0.007 596	0.992 404	57	62,104	1,325	0.021 335	0.978 665
15	96,285	735	0.007 634	0.992 366	58	60,779	1,394	0.022 936	0.977 064
16	95,550	732	0.007 661	0.992 339	59	59,385	1,468	0.024 720	0.975 280
17	94,818	729	0.007 688	0.992 312	60	57,917	1,546	0.026 693	0.973 307
18	94,089	727	0.007 727	0.992 273	61	56,371	1,628	0.028 880	0.971 120
19	93,362	725	0.007 765	0.992 235	62	54,743	1,713	0.031 292	0.968 708
20	92,637	723	0.007 805	0.992 195	63	53,030	1,800	0.033 943	0.966 057
21	91,914	722	0.007 855	0.992 145	64	51,230	1,889	0.036 873	0.963 127
22	91,192	721	0.007 906	0.992 094	65	49,341	1,980	0.040 129	0.959 871
23	90,471	720	0.007 958	0.992 042	66	47,361	2,070	0.043 707	0.956 293
24	89,751	719	0.008 011	0.991 989	67	45,291	2,158	0.047 647	0.952 353
25	89,032	718	0.008 065	0.991 935	68	43,133	2,243	0.052 002	0.947 998
26	88,314	718	0.008 130	0.991 870	69	40,890	2,321	0.056 762	0.943 238
27	87,596	718	0.008 197	0.991 803	70	38,569	2,391	0.061 993	0.938 007
28	86,878	718	0.008 264	0.991 736	71	36,178	2,448	0.067 665	0.932 335
29	86,160	719	0.008 345	0.991 655	72	33,730	2,487	0.073 733	0.926 267
30	85,441	720	0.008 427	0.991 573	73	31,243	2,505	0.080 178	0.919 822
31	84,721	721	0.008 510	0.991 490	74	28,738	2,501	0.087 028	0.912 972
32	84,000	723	0.008 607	0.991 393	75	26,237	2,476	0.094 371	0.905 629
33	83,277	726	0.008 718	0.991 282	76	23,761	2,431	0.102 311	0.897 689
34	82,551	729	0.008 831	0.991 169	77	21,330	2,369	0.111 064	0.888 936
35	81,822	732	0.008 946	0.991 054	78	18,961	2,291	0.120 827	0.879 173
36	81,090	737	0.009 089	0.990 911	79	16,670	2,196	0.131 734	0.868 266
37	80,353	742	0.009 234	0.990 776	80	14,474	2,091	0.144 466	0.855 534
38	79,611	749	0.009 408	0.990 592	81	12,383	1,964	0.158 605	0.841 395
39	78,862	756	0.009 586	0.990 414	82	10,419	1,816	0.174 297	0.825 703
40	78,106	765	0.009 794	0.990 206	83	8,603	1,648	0.191 561	0.808 439
41	77,341	774	0.010 008	0.989 992	84	6,955	1,470	0.211 359	0.788 641
42	76,567	785	0.010 252	0.989 748	85	5,485	1,292	0.235 552	0.764 448
43	75,782	797	0.010 517	0.989 483	86	4,193	1,114	0.265 681	0.734 319
44	74,985	812	0.010 829	0.989 171	87	3,079	933	0.303 020	0.696 980
45	74,173	828	0.011 163	0.988 837	88	2,146	744	0.346 692	0.653 308
46	73,345	848	0.011 562	0.988 438	89	1,402	555	0.395 863	0.604 137
47	72,497	870	0.012 000	0.988 000	90	847	385	0.454 545	0.545 455
48	71,627	896	0.012 509	0.987 491	91	462	246	0.532 468	0.467 534
49	70,731	927	0.013 106	0.986 894	92	216	137	0.634 259	0.365 741
50	69,804	962	0.013 781	0.986 219	93	79	58	0.734 177	0.265 823
51	68,842	1,001	0.014 541	0.985 459	94	21	18	0.857 143	0.142 857
52	67,841	1,044	0.015 389	0.984 611	95	3	3	1.000 000	0.000 000

TABLE IV. TRIGONOMETRIC FUNCTIONS

Angle	Sin	Cos	Tan	Angle	Sin	Cos	Tan
0°	.000	1.000	.000	45°	.707	.707	1.000
1°	.018	.999	.018	46°	.719	.695	1.036
2°	.035	.999	.035	47°	.731	.682	1.072
3°	.052	.998	.052	48°	.743	.669	1.111
4°	.070	.997	.070	49°	.755	.656	1.150
5°	.087	.996	.087	50°	.766	.643	1.192
6°	.105	.994	.105	51°	.777	.629	1.235
7°	.122	.992	.123	52°	.788	.616	1.280
8°	.139	.990	.141	53°	.799	.602	1.327
9°	.156	.988	.158	54°	.809	.588	1.376
10°	.174	.985	.176	55°	.819	.574	1.428
11°	.191	.982	.194	56°	.829	.559	1.483
12°	.208	.978	.213	57°	.839	.545	1.540
13°	.225	.974	.231	58°	.848	.530	1.600
14°	.242	.970	.249	59°	.857	.515	1.664
15°	.259	.966	.268	60°	.866	.500	1.732
16°	.276	.961	.287	61°	.875	.485	1.804
17°	.292	.956	.306	62°	.883	.469	1.881
18°	.309	.951	.325	63°	.891	.454	1.963
19°	.326	.946	.344	64°	.899	.438	2.050
20°	.342	.940	.364	65°	.906	.423	2.144
21°	.358	.934	.384	66°	.914	.407	2.246
22°	.375	.927	.404	67°	.921	.391	2.356
23°	.391	.921	.424	68°	.927	.375	2.475
24°	.407	.914	.445	69°	.934	.358	2.605
25°	.423	.906	.466	70°	.940	.342	2.747
26°	.438	.899	.488	71°	.946	.326	2.904
27°	.454	.891	.510	72°	.951	.309	3.078
28°	.469	.883	.532	73°	.956	.292	3.271
29°	.485	.875	.554	74°	.961	.276	3.487
30°	.500	.866	.577	75°	.966	.259	3.732
31°	.515	.857	.601	76°	.970	.242	4.011
32°	.530	.848	.625	77°	.974	.225	4.331
33°	.545	.839	.649	78°	.978	.208	4.705
34°	.559	.829	.675	79°	.982	.191	5.145
35°	.574	.819	.700	80°	.985	.174	5.671
36°	.588	.809	.727	81°	.988	.156	6.314
37°	.602	.799	.754	82°	.990	.139	7.115
38°	.616	.788	.781	83°	.992	.122	8.144
39°	.629	.777	.810	84°	.994	.105	9.514
40°	.643	.766	.839	85°	.996	.087	11.430
41°	.656	.755	.869	86°	.997	.070	14.300
42°	.669	.743	.900	87°	.998	.052	19.081
43°	.682	.731	.933	88°	.999	.035	28.636
44°	.695	.719	.966	89°	.999	.018	57.290

Answers to Exercises

Note. Answers to odd-numbered problems are given here. Answers to even-numbered problems are furnished free in a separate pamphlet when requested by the instructor.

Exercise 1. Page 5

1. 57; 15; 756; $\frac{12}{7}$. **3.** 5; -35; -300; $-\frac{3}{4}$. **5.** 12; -12; 0; 0.
7. $1 - 2x$. **9.** $-2a - 1$. **11.** $10 - 8x - 3y$.
13. $-7 - 9x$. **15.** $-(2x + 5a - 4)$. **17.** $\frac{3}{5}$. **19.** $\frac{15}{7}$.
21. $\frac{a}{c}$. **23.** $-\frac{12}{5}$. **25.** $\frac{3}{2y}$. **27.** $-\frac{2}{a}$. **29.** $\frac{1}{5}$.
31. $\frac{3}{10}$. **33.** $\frac{5}{14}$. **35.** $\frac{c}{2d}$. **37.** $\frac{6}{7}$. **39.** $\frac{3}{10}$.
41. $\frac{5c}{3}$. **43.** $\frac{4}{35}$. **45.** $\frac{9}{2b}$. **47.** $\frac{13}{20}$. **49.** $\frac{77}{104}$.
51. $\frac{7}{25}$. **53.** $\frac{2}{9d}$. **55.** $\frac{27}{2}$. **57.** $\frac{8}{3}$.

Exercise 2. Page 8

1. 16. **3.** 100,000. **5.** -32. **7.** $\frac{1}{16}$. **9.** $\frac{1}{16}$.
11. $-\frac{1}{8}$. **13.** $\frac{27}{64}$. **15.** 48. **17.** z^{2+n}. **19.** y^{12}.
21. h^3k^3. **23.** a^6b^3. **25.** h^4x^{4m}. **27.** d^3. **29.** $\frac{1}{x^5}$.
31. a^4. **33.** $\frac{a^2}{16}$. **35.** $\frac{81}{a^4}$. **37.** $25y^2$. **39.** $-8h^9$.
41. $\frac{9a^2}{16b^4}$. **43.** $\frac{c^6x^3}{125a^3}$. **45.** $36a^5y^7$. **47.** $-12b^5c^2$. **49.** $6a^{3+h}b^3$.
51. $3w^2x$. **53.** $\frac{1}{4a^3y}$. **55.** $-\frac{5x^2}{y^2}$. **57.** $\frac{16y^2}{3x}$. **59.** $-\frac{9x^3}{4w^4}$.
61. 10. **63.** $\frac{2}{3}$. **65.** y^3. **67.** $2z^6$.
69. $\frac{1}{5}$. **71.** a. **73.** $2x^4$. **75.** $8a^2y^3$.
77. $\frac{5}{x}$. **79.** $\frac{y^3}{7}$. **81.** $\frac{z^2}{y^4}$. **83.** $\frac{5}{3x}$.
85. $\frac{3z^2}{2x^3}$. **87.** $\frac{9a}{y^2z}$. **89.** $\frac{3x^3}{2y^2z}$. **91.** $\frac{10xy^2}{7w^3}$.

Exercise 3. Page 11

1. $6x^2 - 14x$. **3.** $15y - 30xy - 10y^3$.
5. $8a^3 + 1$. **7.** $5a^4 - 17a^3 + 13a^2 + 5a - 6$.
9. $y^5 + 3y^2 + 5y - 3$. **11.** $-3a^2 + 6b$.
13. $-\frac{2x}{y^2} + \frac{5}{2y} + \frac{8}{x^2}$. **15.** $3a + 2 + \frac{5}{2a - 1}$.

17. $x^3 - x^2 - 4$. **19.** $2 - 5y$. **21.** $z^2 + 3z + 9$.
23. $x^4 - 2x^3 + 4x^2 - 8x + 16$. **25.** $15a - 20v$.
27. $-2hx + 2hy$. **29.** $a^2 - 2ax + x^2$.
31. $4h^2 - 12hk + 9k^2$. **33.** $a^2 - 2ab + b^2$.
35. $4 - 9x^2$. **37.** $8 - 6a + a^2$. **39.** $-10 + 19x - 6x^2$.
41. $4x^2 - 25y^2$. **43.** $9x^2 - 16a^2y^2$. **45.** $x^2 - \frac{1}{4}$.
47. $x^2 - 4x^3w + 4x^4w^2$. **49.** $\frac{1}{9} - \frac{2}{3}z + z^2$.
51. $9a^2 + 18ab + 9b^2$. **55.** $c^2 + 2cy + y^2 - 4$.
57. $a^2 - 2ab + b^2 + 10a - 10b + 25$. **59.** $4a^2 + 12ab - 4ax + 9b^2 - 6bx + x^2$.
61. $16a^2 + b^2 + c^2 - 8ab - 8ac + 2bc$. **63.** $9a^2 - y^2 + 8y - 16$.
65. $c^2 - 4cd + 4d^2 - a^2 - 2ax - x^2$.
67. $4a^2 + 9b^2 + 16c^2 + 12ab + 16ac + 24bc$.
69. $9a^2 + 4b^2 + 9c^2 - 12ab + 18ac - 12bc$.

Exercise 4. Page 14

1. $a(x + y)$. **3.** $x(b - a)$. **5.** $t(t - ct^2 - 4a)$. **7.** $(2x - y)(2x + y)$.
9. $(2a + 3b)(2a - 3b)$. **11.** $(3z + \frac{1}{2})(3z - \frac{1}{2})$. **13.** $(5w + cd)(5w - cd)$.
15. $(w + 6)^2$. **17.** $(x + \frac{1}{2})^2$. **19.** $(2x + 3z)^2$.
21. $(ab - 3x)^2$. **23.** $(x + 5)(x + 3)$. **25.** $(x - 3)(x + 2)$.
27. $(2x + 1)(x + 3)$. **29.** $(3a + 4)(5a - 2)$. **31.** $(x + y)(3a - 5b)$.
33. $(x + y)(b + 2h)$. **35.** $(x - b)(4h - 5c)$. **37.** $(x - 2)(x - 1)(x + 1)$.
39. $(x - y)(x + y)(b + c)$. **41.** $(c - 7a - 7b)(c + 7a + 7b)$.
43. $(6a - 5b)(a - b)$. **45.** $(8a - 3c)^2$.
47. $r(5h - 2)(3h - 1)$. **49.** $(5x^2 - 1)(x^2 - 3)$.
51. $2c(2a - 3c)(2a + 3c)$. **53.** $(9c^2 + 4d^2)(3c + 2d)(3c - 2d)$.
55. $(2a + 3 - 5x)(2a + 3 + 5x)$. **57.** $(2h - 3a + 2)(2h + 3a - 2)$.
59. $(y + z - 2x)(y + z + 2x)$. **61.** $(2a - 3z - 1)(2a + 3z + 1)$.
63. $(5 - 3z - 3w)(3 + 2z + 2w)$. **65.** $(a + 1)^4$.
67. $2(5x + 5y - 3w)(2x + 2y - w)$. **71.** $(z^2 - z + 1)(z^2 + z + 1)$.
73. $(z^2 - 2z - 1)(z^2 + 2z - 1)$. **75.** $(a^2 - ay - 4y^2)(a^2 + ay - 4y^2)$.
77. $(w^2 - 2wx + 2x^2)(w^2 + 2wx + 2x^2)$.

Exercise 5. Page 17

1. $a^2 + ah + h^2$. **3.** $x^3 + x^2y + xy^2 + y^3$. **5.** $c^3 + w^3$. **7.** $27a^3 - c^3$.
9. $(d - y)(d^2 + dy + y^2)$. **11.** $(1 - v)(1 + v + v^2)$.
13. $(z + 10)(z^2 - 10z + 100)$. **15.** $(6x - yz)(36x^2 + 6xyz + y^2z^2)$.
17. $c^3 + 3c^2d + 3cd^2 + d^3$. **19.** $125 - 75y + 15y^2 - y^3$.
21. $y^3 - 9xy^2 + 27x^2y - 27x^3$. **23.** $c^3 + 9b^2c^2 + 27b^4c + 27b^6$.
25. $a^4 - a^3y + a^2y^2 - ay^3 + y^4$. **27.** $x^4 + x^3 + x^2 + x + 1$.
29. $a^6 + 2a^4b + 4a^2b^2 + 8b^3$. **31.** $(a - c)(a^4 + a^3c + a^2c^2 + ac^3 + c^4)$.
33. $(y^2 + 9)(y - 3)(y + 3)$. **35.** $(h + k)(h^2 - hk + k^2)(h^6 - h^3k^3 + k^6)$.
37. $(a - 3x^2)(a^2 + 3ax^2 + 9x^4)$. **39.** $(2x^2 - 2x + 1)(2x^2 + 2x + 1)$.
41. Prime. **43.** $(4x^2 + y^4)(2x - y^2)(2x + y^2)$.

Exercise 6. Page 19

1. $5a$. 3. $-\frac{3}{a}$. 5. $-\frac{2}{3}$. 7. $\frac{cd}{2}$. 9. $\frac{x+2}{2}$.

11. $\frac{4x-4y}{c}$. 13. $\frac{m-3}{m-4}$. 15. $\frac{x-a}{x+2a}$. 17. $\frac{a+x}{2a+x}$.

19. $\frac{a}{3x^2-3xy+3y^2}$. 21. $\frac{9bx-10ay}{6ab}$. 23. $\frac{3a^2-20c+24c^2+4ac}{4ac}$.

25. $\frac{8x^2+8}{(2x+3)(3x-2)}$. 27. $\frac{12a-6}{a-2}$. 29. $\frac{x^2-17x+1}{x^2+x-12}$.

31. $\frac{2}{(n+4)(1-n)}$. 33. $\frac{ax+12a}{3x(4-x)(4+x)}$. 35. $\frac{13+39x+25x^2+12x^3}{27-8x^3}$.

Exercise 7. Page 22

1. $\frac{c}{10d}$. 3. $\frac{x}{5d^3}$. 5. $\frac{hw}{3a}$. 7. $\frac{(x-1)(x+4)}{x}$.

9. $\frac{ck}{aw}$. 11. $\frac{(h+3)(x+y)}{3(3-h)}$. 13. $\frac{2(x-1)^2}{15(4x-1)}$.

15. $\frac{(a-x)(5a-x)}{b}$. 17. $\frac{29}{77}$. 19. $\frac{36x^2-4}{36y^2-9}$.

21. $\frac{1}{a}$. 23. $\frac{b-a}{b}$. 25. $\frac{x^2}{y(3xy+2)}$. 27. $ab(a^2b^2+1)$.

29. $\frac{9}{2x-3a}$. 31. $3a(3a-2)$. 33. $4c^2d(c^2+3cd+9d^2)$.

35. $\frac{2a-3b}{a^2b^2}$. 37. $\frac{1}{75}$. 39. $-\frac{5}{63}$.

41. $\frac{1}{5d-3a}$. 43. $\frac{6a+2c}{5a-3b}$. 45. $\frac{x(6x^2-5xy-15y^2)}{3x-7y}$.

47. $\frac{5}{a+3}$. 49. $\frac{(x+1)(2x^2+x+1)}{x^2(2x^2+x)}$. 51. $\frac{(1+a)(13a-3)}{2a(5a-1)}$.

Exercise 8. Page 27

1. $\frac{5}{6}$. 3. 0. 5. $-\frac{5}{7}$. 7. $-\frac{2}{5}$. 9. .11. 11. 3.

13. 2. 15. -3. 17. $-\frac{5}{3}$. 19. $\frac{3}{2}$. 21. $\frac{3h+5a}{c}$.

23. $\frac{2b}{3-a}$. 25. $\frac{5c}{b-2a}$. 27. $\frac{a+1}{4}$. 29. $b-ab$. 31. $\frac{c-d}{a+b}$.

33. $y=\frac{7+3x}{2x+5}$; $x=\frac{7-5y}{2y-3}$. 35. $y=\frac{3x+4}{2x-2}$; $x=\frac{2y+4}{2y-3}$.

37. 5. 39. -1. 41. $-\frac{74}{27}$. 43. 2. 45. -5.

47. $a=\frac{f}{m}$. 49. $v=\frac{s-k}{t}$. 51. $-2n$. 53. $\frac{12}{35}$. 55. 283.46

Exercise 9. Page 29

1. $32\frac{1}{2}''$; $35\frac{1}{2}''$. **3.** 13 nickels; 39 dimes; 36 quarters.

5. $4\frac{12}{19}$ da. **7.** 20 gal. **9.** $9:36\frac{12}{23}$ A.M. **11.** $63\frac{1}{3}$ lb.

13. 490 lb. **15.** \$42,857.14. **17.** \$1000. **19.** \$5000.

21. 15 mi. per hr. **23.** 20 yr. **25.** 200 mi. per hr. **27.** $3\frac{3}{8}$ hr.

29. \$8300 at 5%; \$3700 at 3% loss **31.** $10\frac{10}{11}$ min.

Exercise 10. Page 35

1. x^{15}. **3.** $16a^{12}$. **5.** $.09c^2d^6$.

7. $\frac{a^2}{b^2}$. **9.** $\frac{z^{3n}}{a^{3h}}$. **11.** ± 8; ± 7; ± 9; $\pm \frac{1}{3}$; $\pm \frac{2}{5}$; $\pm .1$.

13. 2; 3; -1; -6; $\frac{1}{3}$; 5. **15.** b. **17.** 3.

19. 26. **21.** $2x$. **23.** -4. **25.** 6. **27.** 2.

29. 5. **31.** -1. **33.** 20. **35.** $-\frac{1}{2}$. **37.** $\frac{1}{5}$.

39. $\frac{2}{9}$. **41.** y^2. **43.** $2x$. **45.** $3a^2$. **47.** $\frac{x}{2}$.

49. $\frac{ab}{3}$. **51.** $3w^2$. **53.** $3x^4$. **55.** $\frac{5}{7}$. **57.** $2xy^2$.

59. .5. **61.** $\frac{1}{2}x$. **63.** $.2x^5$. **65.** $-\frac{3}{10}$.

67. $\frac{5x^2}{2z^6}$. **69.** $\frac{x^2}{y}$. **71.** $\frac{4}{ay^2}$.

Exercise 11. Page 38

1. 5. **3.** $\frac{1}{4}$. **5.** $\frac{1}{125}$. **7.** $\frac{1}{4}$. **9.** $\frac{3}{2}$.

11. -2. **13.** 8. **15.** $\frac{16}{9}$. **17.** .6. **19.** 1.

21. -5. **23.** 4. **25.** 32. **27.** $\frac{8}{27}$. **29.** 9.

31. $\frac{1}{b^4}$. **33.** $\frac{3}{h^4}$. **35.** $\frac{a}{4x^3}$. **37.** a^3d^2.

39. $\frac{3a^3}{2x^5}$. **41.** $5y^{-4}$. **43.** $3^{-1} \cdot 4a^2x^{-1}y^{-3}$. **45.** $c(x-5y)^{-1}$.

47. $\sqrt[7]{a^5}$. **49.** $a\sqrt[4]{x}$. **51.** $a^{\frac{8}{5}}$. **53.** $\sqrt{2xy}$.

55. $(a+b)^{\frac{2}{3}}$. **57.** x^4. **59.** 1. **61.** $\frac{125}{a^6}$.

63. $x^{\frac{2}{15}}$. **65.** $\frac{1}{b^6}$. **67.** $\frac{1}{a^6x^9}$. **69.** $\frac{9y^6}{x^2}$.

71. $\frac{a^6}{25b^4}$. **73.** 625. **75.** $\frac{x^4}{16}$. **77.** $\frac{x^3}{125}$.

79. $\frac{1}{a^{\frac{4}{3}}}$. **81.** $\frac{x^{\frac{5}{2}}}{y^{\frac{2}{3}}}$. **83.** $\frac{a^{\frac{1}{12}}}{2y^{\frac{1}{4}}}$. **85.** $\frac{1}{x^2y^7}$.

87. $\frac{8a^{\frac{15}{2}}x^3}{27}$. **89.** $9ax^2$. **91.** $9m^4$. **93.** x^3y^5.

95. $\dfrac{a+b}{ab}$. 97. $\dfrac{b^3-a^3}{a^3b^3}$. 99. $\dfrac{1-a^2b}{1+a^2b}$. 101. $\dfrac{4a}{a+4}$.

103. $x^{-2}+y^{-2}$. 105. $x^{\frac{1}{2}}-y^{\frac{1}{2}}$. 107. $a^{\frac{2}{3}}+2a^{\frac{1}{3}}b+b^2$.

109. $x^8-4x^4y^{-1}+4y^{-2}$. 111. $a^3-b^{\frac{3}{2}}$.

113. $(x-y^{-1})(x+y^{-1})$. 115. $(2a^{\frac{1}{2}}-3b^{\frac{1}{2}})(2a^{\frac{1}{2}}+3b^{\frac{1}{2}})$.

117. $(x-y^{-1})^2$. 119. $(2a^{\frac{1}{3}}-5b^{\frac{1}{3}})^2$.

121. $x^{-2}-y^{-2}$. 123. $a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}$.

Exercise 12. Page 41

1. $2\sqrt{5}$ or 4.472. 3. $3\sqrt{3}$ or 5.196. 5. $.3\sqrt{5}$ or .6708.
7. $3\sqrt[3]{4}$ or 4.761. 9. $-3\sqrt[3]{2}$ or -3.780. 11. $a^4\sqrt{a}$.
13. $x^2\sqrt[4]{x^2}$. 15. $2a\sqrt[3]{a}$. 17. $2ay^2\sqrt{3ay}$.
19. $3y\sqrt[3]{2y^2}$. 21. $-3a^2\sqrt[3]{a^2}$. 23. $xy^2\sqrt[4]{3y^2}$.
25. $2ay^4\sqrt[4]{2}$. 27. $.5x^3\sqrt{x}$. 29. $\dfrac{x}{2y^2}\sqrt[3]{5x^2}$.
31. $-\dfrac{2}{ab^2}\sqrt[3]{2}$. 33. $x\sqrt[3]{27-z^3}$. 35. $2a^{2n}$.
37. $\dfrac{1}{xy}\sqrt{cx+2dy^2}$. 39. $8\sqrt{2}$. 41. $8\sqrt[3]{3}$.
43. $(3-x)\sqrt[3]{3x}$. 45. $\sqrt{6}$ or 2.449. 47. $3\sqrt{5}$ or 6.708.
49. 45. 51. $-2\sqrt[3]{9}$ or -4.160. 53. $\sqrt{3}$ or 1.732.
55. $\sqrt[3]{9}$ or 2.080. 57. $\sqrt{5}$ or 2.236. 59. $\dfrac{h}{2}$.
61. $3x\sqrt{2x}$. 63. $32a$. 65. $7\sqrt{5}-9$.
67. $18+13\sqrt{6}$. 69. $27+10\sqrt{2}$. 71. $14-4\sqrt{6}$.
73. $-xyz\sqrt[5]{z^4}$. 75. $\sqrt{18a}$. 77. $\sqrt[3]{27b^2}$.

Exercise 13. Page 43

1. $\frac{1}{3}\sqrt{3}$ or .577. 3. $\frac{1}{9}\sqrt{6}$ or .272. 5. $-\frac{1}{10}\sqrt[3]{70}$ or $-.4121$.
7. $\frac{1}{10}\sqrt[3]{100}$ or .4642. 9. $\frac{1}{50}\sqrt{30}$ or .1095. 11. $\frac{1}{3}\sqrt{3}$ or .577.
13. $\frac{6}{5}\sqrt{5}$ or 2.683. 15. $\frac{2}{3}\sqrt{3}$ or 1.155. 17. $\frac{1}{3}\sqrt{15}$ or 1.291.
19. $\frac{2}{5}\sqrt{15}$ or 1.549. 21. $\frac{3}{2}\sqrt[3]{2}$ or 1.890. 23. $\frac{1}{6}(9-5\sqrt{3})$ or .057.
25. $\frac{1}{5}(4-6\sqrt{2}-\sqrt{6}+3\sqrt{3})$ or $-.347$. 27. $\frac{1}{43}(17+4\sqrt{10})$ or .689.
29. $\frac{1}{3}\sqrt{3x}$. 31. $\dfrac{\sqrt{3}}{3x}$. 33. $\dfrac{\sqrt[3]{4ab^2}}{2b}$.
35. $\dfrac{\sqrt[3]{3a^2b}}{3a^2}$. 37. $\dfrac{\sqrt[4]{2a}}{2}$. 39. $\frac{1}{7}\sqrt[3]{14}$ or .344.
41. $\dfrac{\sqrt{10ab}}{10ab}$. 43. $-\dfrac{\sqrt[3]{25}}{5ab}$. 45. $\dfrac{2\sqrt{h(a+x)}}{a+x}$. 47. $\dfrac{\sqrt{x}}{x^3}$.

49. $\frac{\sqrt[3]{x}}{x}$.
51. $\frac{\sqrt{5x}}{5x^2}$.
53. $\frac{\sqrt{5b(5a^2 - 9b)}}{5ab}$.

55. $\frac{a + \sqrt{ac}}{a - c}$.
57. $\frac{1}{4}(2\sqrt{3} + \sqrt{30} - 3\sqrt{2})$ or 1.175.

59. $\frac{\sqrt[k]{2^{k-1}ab^{k-3}}}{2b}$.
61. $\frac{x^3\sqrt[m]{2 \cdot 5^{m-1}a^{m-2}y^{m-3}}}{5ay}$.

Exercise 14. Page 45

1. $\sqrt[6]{a^2b^3}$.
3. $5\sqrt[4]{a^3}$.
5. $a\sqrt[3]{b^2}$.
7. $2\sqrt[6]{a^4b^3}$.
9. $a\sqrt[4]{x^2y^3}$.
11. $2a^2b^2\sqrt[6]{b^3c^5}$.
13. $\sqrt{x}$.
15. $\sqrt[3]{y^2}$.
17. $\sqrt[3]{z}$.
19. $\sqrt{3}$.
21. $\sqrt[4]{6}$.
23. $\sqrt{2}$.
25. $\sqrt{3a}$.
27. $\sqrt{3b}$.
29. c^2.
31. $\sqrt[3]{x^2}$.
33. $2\sqrt{2}$.
35. $25\sqrt{5}$.
37. $4x^2\sqrt{2x}$.
39. $48\sqrt[3]{3}$.
41. $\sqrt[10]{z}$.
43. $\sqrt[4]{3}$.
45. $\sqrt[16]{x}$.
47. $\sqrt[4]{a}$.
49. $\sqrt[12]{y^7}$.
51. $a\sqrt[6]{a}$.
53. $\frac{\sqrt[4]{b^3}}{b}$.
55. $\sqrt[12]{5}$.
57. $\frac{1}{3}\sqrt[4]{27}$.
59. $\frac{\sqrt[4]{c^3d}}{c}$.
61. $\frac{1}{5}\sqrt{10}$.
63. $-\frac{x\sqrt[3]{2x^2}}{2}$.
65. $\frac{\sqrt{x}}{x^3}$.
67. $\sqrt[4]{2a}$.
69. $3\sqrt[3]{2x}$.
71. $3\sqrt[12]{3}$.
73. $\frac{a\sqrt[3]{ab^2}}{b}$.
75. $\frac{y\sqrt{6xy}}{2b^2}$.
77. $a^2x^4\sqrt[3]{ax^2}$.
79. $\sqrt[6]{2}$.
81. $2a^5\sqrt[3]{4}$.
83. $a\sqrt{2}$.
85. $\frac{(b^2 - 2x)\sqrt[3]{x}}{2x}$.
87. $2\sqrt{3} + 2\sqrt{2}$.
89. $\frac{3a + \sqrt{3a} + \sqrt{3a(a+b)} + \sqrt{a+b}}{b - 2a}$.
91. $(2x - b)\sqrt[3]{a}$.
93. $(a - z)\sqrt{3x}$.
95. $\frac{(a+2)\sqrt{a^2 - 1}}{a(a+1)}$.

Exercise 15. Page 48

1. $<$.
3. $<$.

Exercise 17. Page 52

1. $7;\ -3;\ \frac{5}{2};\ \frac{21}{5}$.
3. $9;\ b^4 - b^2 + 3;\ \frac{c^2 - cd + 3d^2}{d^2};\ 4x^2 - 6x + 5$.
5. $-12;\ \frac{s+2}{2(1-s)};\ \frac{c^2+2}{1-c^2};\ \frac{3x+2y}{y-3x};\ \frac{(a+2)(1-b)}{(1-a)(b+2)}$.
7. $0;\ 3a + 4b$.

Exercise 18. Page 54

15. (a) $y = 6$; $y = -4$. **17.** $y = x$.
19. $x - y = -2$. **23.** Slope $= -\frac{1}{2}$.
25. Slope $= 3$. **27.** Slope $= -\frac{3}{2}$; y-intercept $= \frac{9}{2}$.
29. Slope $= -\frac{3}{8}$; y-intercept $= 0$. **31.** Parallel.

Exercise 19. Page 56

Note 1. In the solutions of systems of equations in this answer book, values of the unknowns will be arranged in their natural alphabetical order.

1. (3, 2). **3.** $(-1, -3)$. **5.** $(0, -4)$. **7.** (2, 2).
9. (0, 0). **11.** $(-\frac{5}{2}, \frac{3}{2})$. **13.** $(-\frac{5}{4}, -\frac{1}{4})$. **15.** $(-\frac{37}{21}, -\frac{12}{7})$.
17. (.7, 1.2). **19.** $(5, -3)$. **21.** $\left(\dfrac{3h + k}{9h + 2k}, -\dfrac{hk}{9h + 2k}\right)$.
23. $\left(\dfrac{2b - 3a}{b^2 - a^2}, \dfrac{2a - 3b}{b^2 - a^2}\right)$. **25.** $(a + b, b)$. **27.** $\left(\dfrac{c}{d}, \dfrac{d}{2c}\right)$.
29. Inconsistent. **31.** Dependent. **33.** Dependent.
35. 1st, 3 lb.; 2d, 6 lb. **37.** 40 lb. silver; 80 lb. lead. **39.** 64.

Exercise 20. Page 59

1. $(1, 2, -2)$. **3.** $(-\frac{1}{27}, -\frac{2}{3}, \frac{4}{3})$. **5.** $(\frac{1}{2}, 3, 2)$. **7.** $(-2, 3, 3)$.
9. $(\frac{1}{2}, -\frac{3}{2})$. **11.** $(2, -\frac{1}{2}, -\frac{1}{3})$. **13.** 465.

Exercise 21. Page 61

19. (0, 0).

Exercise 22. Page 62

1. -9. **3.** -42.
5. $d_1b_2c_3 + d_2b_3c_1 + d_3b_1c_2 - d_3b_2c_1 - d_1b_3c_2 - d_2b_1c_3$.

Exercise 23. Page 63

9. $\left(\dfrac{b + c - a}{2a}, \dfrac{a + c - b}{2b}, \dfrac{a + b - c}{2c}\right)$.

Exercise 24. Page 65

1. $5i$. **3.** $i\sqrt{23}$. **5.** $\frac{1}{3}i$. **7.** $\dfrac{i\sqrt{6}}{2}$.
9. $.3i$. **11.** $2ix\sqrt{2}$. **13.** $8ix^2y^2\sqrt{2}$. **15.** $\pm 9i$.
17. $\pm \frac{3}{4}i$. **19.** $\pm \frac{2}{7}i\sqrt{7}$. **21.** $-i$. **23.** -1. **25.** -1.
27. 10. **29.** 13. **31.** $7 + 24i$. **33.** $7 - 24i$. **35.** $4 + 19i$.
37. $(10 + 3i)\sqrt{2}$. **39.** $(4i - 19)$; $(-34 - 6i)$; $(-66 - 70i)$.

Exercise 25. Page 67

1. ± 5.
3. $\pm 3i$.
5. $\pm \frac{3}{2}i$.
7. $\pm \frac{1}{2}\sqrt{c}$.
9. $\pm \frac{1}{2}\sqrt{2}$.
11. $\pm \frac{1}{3}\sqrt{30}$.
13. $\pm \dfrac{\sqrt{a(d+c)}}{2a}$.
15. $5;\ -2$.
17. $\frac{1}{2};\ -3$.
19. $0;\ \frac{9}{5}$.
21. $0;\ \frac{3}{2}$.
23. $\frac{3}{4};\ \frac{1}{2}$.
25. $-4;\ \frac{3}{5}$.
27. $\frac{4}{5};\ \frac{2}{3}$.
29. $\frac{5}{4};\ -\frac{3}{2}$.
31. $1;\ -\frac{4}{9}$.
33. $0;\ -\dfrac{c}{3b}$.
35. $-\dfrac{3b}{2};\ b$.
37. $\dfrac{3b}{2a};\ -\dfrac{b}{a}$.
39. $-3;\ \frac{5}{2};\ -\frac{7}{3}$.

Exercise 26. Page 70

1. $-7;\ 1$.
3. $\frac{1}{3}(-1 \pm \sqrt{2})$: $.138;\ -.805$.
5. $3 \pm 2i$.
7. $\frac{5}{2};\ \frac{5}{2}$.
9. $\frac{1}{2}(2 \pm i\sqrt{10})$.
11. $-\frac{1}{2}b;\ 3b$.
13. $\dfrac{-k \pm \sqrt{k^2 - 4hP}}{2h}$.
15. $1;\ -\frac{5}{3}$.
17. $1 \pm 3i$.
19. $\frac{3}{2};\ \frac{3}{2}$.
21. $\frac{1}{3}(-1 \pm \sqrt{2})$: $.138;\ -.805$.
23. $\frac{1}{2}(4 \pm \sqrt{10})$: $.419;\ 3.581$.
25. $\frac{1}{2}(1 \pm 2i\sqrt{3})$.
27. $\frac{5}{6};\ -\frac{8}{3}$.
29. $-\dfrac{2d}{3};\ \dfrac{3d}{2}$.
31. $\dfrac{3k \pm \sqrt{9k^2 - 120k}}{10k}$.
33. $-\dfrac{5}{3};\ \dfrac{2}{1+h}$.
35. $\frac{3}{4};\ \frac{3}{4}$.
37. $\frac{1}{3}(1 \pm \sqrt{19})$: $1.786;\ -1.120$.
39. $\frac{1}{5}(2 \pm \sqrt{5})$: $.847;\ -.047$.
41. $0;\ \frac{7}{6}$.
43. $(6 \pm \sqrt{41})$: $12.403;\ -.403$.
45. $-\frac{5}{2};\ 4$.
47. $\dfrac{-a \pm \sqrt{a^2 + 6gS}}{g}$.
49. $-\frac{1}{3}h;\ -k$.
51. $\frac{1}{3}(2h - 1);\ \frac{1}{5}(y + 2)$.
53. $14;\ 31$.
55. 8.138 rd.
57. 43.8 mi. per hr.
59. $A \pm .6\sqrt{A}$; 17.32 sq. ft.

Exercise 27. Page 73

1. Vertex, $(0, 0)$; axis, $x = 0$; min. $= 0$.
3. Vertex, $(2, 3)$; axis, $x = 2$; min. $= 3$.
5. Vertex, $(-1, 8)$; axis, $x = -1$; max. $= 8$.
7. Vertex, $(0, 5)$; axis, $x = 0$; min. $= 5$.
9. $2;\ -4$.
11. No real roots.
13. $1.6;\ -4.1$.
15. Min. $= -1$.
17. Max. $= 17$.
19. Vertex, $(2, 0)$; axis, $y = 0$.
21. Parabola with axis parallel to x-axis, concave in direction of positive x-axis if $a > 0$, etc. Value of y at vertex is $-b/2a$.
23. $30;\ 30$.
25. $7\frac{1}{2}''$ by $15''$.

Exercise 28. Page 76

1. Disc. = 9; real, unequal, and rational.
3. Disc. = − 59; imaginary and unequal.
5. Disc. = 64; real, unequal, and rational.
7. Disc. = 0; real, equal, and rational.
9. Disc. = − 60; imaginary and unequal.
11. Roots imaginary.
13. Disc. = 0; graph is tangent to x-axis, concave upward and has axis perpendicular to x-axis.
19. $\pm \frac{4}{3}$.
21. $0;\ -\frac{12}{13}$.
23. $0;\ 5$.

Exercise 29. Page 78

1. Sum = − 5; prod. = − 3.
3. Sum = $-\frac{3}{4}$; prod. = $-\frac{7}{4}$.
5. Sum = $-\frac{7}{9}$; prod. = $-\frac{5}{9}$.
7. $x^2 + 4x - 21 = 0$.
9. $28x^2 + 41x + 15 = 0$.
11. $x^2 + 9 = 0$.
13. $x^2 + 4x - 41 = 0$.
15. $2x^2 - 4x - 7 = 0$.
17. $x^2 - 2ax + a^2 + b^2 = 0$.
19. No factors of the specified type. (Why?)
21. $\frac{5}{7}$.
23. $-\frac{1}{2}$.
25. $h = -\frac{1}{5}$.
27. $h = -\frac{1}{17}$.
29. $h = \pm 3\sqrt{5}$.
31. $(4x + 9)(3x - 4)$.
33. $(9x - 8y)(3x + 2y)$.
35. $(2x - 3 + \sqrt{2})(2x - 3 - \sqrt{2})$.
41. $h = 2$.
43. $h = 0;\ k = -\frac{1}{3}$.

Exercise 30. Page 80

1. $\pm 3;\ \pm 2$.
3. $\pm \frac{1}{2}\sqrt{2};\ \pm 3i$.
5. $\pm \frac{1}{2};\ \pm \frac{1}{2}$.
7. $\pm \frac{3}{2};\ \pm \frac{3}{2}i$.
9. $\pm 1;\ \pm \frac{3}{2}$.
11. $\pm \frac{1}{5}\sqrt{5};\ \pm \sqrt{2}$.
13. $1;\ 2;\ -2;\ -3$.
15. $1;\ -3;\ \frac{1}{3}(-3 \pm \sqrt{6})$.
17. $2;\ -1;\ \frac{1}{2}(-5 \pm \sqrt{33})$.
19. $\pm \frac{3}{5};\ \pm \frac{3}{5}i$.
21. $\frac{5}{2};\ \frac{1}{4}(-5 \pm 5i\sqrt{3})$.
23. $-\frac{3}{5};\ \frac{1}{10}(3 \pm 3i\sqrt{3})$.
25. $(\pm 2, \pm 2i);\ (\pm 3, \pm 3i);\ (\pm \frac{1}{5}, \pm \frac{1}{5}i)$.

Exercise 31. Page 82

1. 22.
3. − 15.
5. $\frac{3}{8}\sqrt{5}$.
7. $\frac{9}{4}$.
9. $-3;\ 27$.
11. 2.
13. No solution.
15. $-1;\ \frac{1}{2}$.
17. 0.
19. $4;\ \frac{4}{9}$.
21. $s = \dfrac{v^2}{2g};\ g = \dfrac{v^2}{2s}$.
23. a.
25. $\frac{4}{25}$.
27. $\frac{1}{16}$.
29. $1;\ \frac{9}{25}$.
31. $3;\ -5$.
33. 4.
35. $-\frac{76}{3}$.
37. $\sqrt[3]{4}$.

Exercise 32. Page 87

1. (4; .5); (− 2.8, 2.9).
3. (− 4.1, − 7.1); (2.1, − .9).
5. (1, ± 1.1); (− 1, ± 1.1).
7. (3.2, 3.7).
9. No real solution.

Exercise 33. Page 88

1. $(4, 3)$; $(5, 0)$.

3. $[\frac{1}{2}(2 + i\sqrt{3}), \frac{1}{2}(6 - i\sqrt{3})]$; $[\frac{1}{2}(2 - i\sqrt{3}), \frac{1}{2}(6 + i\sqrt{3})]$.

5. $(-2, -1)$; $(-6, 3)$. **7.** $(2, 1)$; $(\frac{3}{2}, \frac{4}{3})$. **9.** $(2, 5)$; $(2, 5)$.

11. $[\frac{1}{2}(a - b), (a + b)]$; $[\frac{1}{2}(a + b), (a - b)]$.

Exercise 34. Page 89

1. $(1.837, \pm .790)$; $(-1.837, \pm .790)$.

3. $(\frac{4}{3}\sqrt{2}, \pm \frac{4}{3}i)$; $(-\frac{4}{3}\sqrt{2}, \pm \frac{4}{3}i)$.

5. $(\sqrt{5}, \pm 1)$; $(-\sqrt{5}, \pm 1)$. **7.** $(\pm \sqrt{2}, \sqrt{3})$; $(\pm \sqrt{2}, -\sqrt{3})$.

9. $(\frac{1}{2}\sqrt{3}, \pm \frac{1}{2}\sqrt{3})$; $(-\frac{1}{2}\sqrt{3}, \pm \frac{1}{2}\sqrt{3})$. **11.** $(\sqrt{3}, \pm i\sqrt{7})$; $(-\sqrt{3}, \pm i\sqrt{7})$.

Exercise 35. Page 90

1. $(\sqrt{2}, -\sqrt{2})$; $(-\sqrt{2}, \sqrt{2})$; $(\frac{4}{5}\sqrt{5}, \frac{2}{5}\sqrt{5})$; $(-\frac{4}{5}\sqrt{5}, -\frac{2}{5}\sqrt{5})$.

3. $(\frac{1}{2}, 1)$; $(-\frac{1}{2}, -1)$; $(-\sqrt{2}, \frac{1}{3}\sqrt{2})$; $(\sqrt{2}, -\frac{1}{3}\sqrt{2})$.

5. $(14, -4)$; $(-4, -1)$; $(-14, 4)$; $(4, 1)$.

7. $(\frac{3}{2}, -\frac{7}{2})$; $(\frac{5}{2}, -\frac{1}{2})$; $(-\frac{3}{2}, \frac{7}{2})$; $(-\frac{5}{2}, \frac{1}{2})$.

9. $(\frac{3}{2}\sqrt{2}, \frac{7}{2}\sqrt{2})$; $(-\frac{3}{2}\sqrt{2}, -\frac{7}{2}\sqrt{2})$; $(2, 5)$; $(-2, -5)$.

11. $(2, 3)$; $(2, -\frac{4}{5})$; $(\frac{2}{5}, \frac{3}{5})$; $(\frac{5}{3}, -\frac{2}{3})$.

13. 81. **15.** $(\frac{1}{4}, -\frac{1}{7})$; $(\frac{1}{7}, -\frac{1}{4})$. **17.** $(\frac{1}{2}, \pm 1)$; $(-\frac{1}{2}, \pm 1)$.

Exercise 36. Page 92

1. $(-\frac{3}{2}, 5)$; $(\frac{5}{2}, -3)$. **3.** $(\frac{4}{3}\sqrt{15}, \frac{2}{3}\sqrt{15})$; $(-\frac{4}{3}\sqrt{15}, -\frac{2}{3}\sqrt{15})$.

5. $(-1, 3)$; $(6, -4)$; $(3, -5)$; $(-2, 0)$.

7. $(\frac{1}{2}, 2)$; $(-\frac{1}{2}, -2)$; $(1, 1)$; $(-1, -1)$. **9.** $(8, -4)$; $(-2, 1)$.

11. $(\pm \frac{3}{2}, 2, -1)$; $(\pm \frac{3}{2}, 2, 1)$; $(\pm \frac{3}{2}, -2, -1)$; $(\pm \frac{3}{2}, -2, 1)$.

13. ± 25. **15.** $c = \pm \sqrt{9 + 4m^2}$. **17.** $c = \pm \sqrt{a^2 + b^2m^2}$.

19. $(3, 1)$; $(-3, -1)$; $(-1, -3)$; $(1, 3)$.

Exercise 37. Page 93

5. $-\frac{5}{4}$; $\frac{3}{2}$. **7.** $\frac{1}{6}(1 \pm i\sqrt{59})$.

9. Real, unequal, and irrational; -5 and -2.

11. Real, equal, and rational; 5 and $\frac{25}{4}$.

13. $8i$. **15.** $\frac{27}{4}$; 0. **17.** ± 2. **19.** $20x^2 + 7x - 6 = 0$.

21. $x^2 - 6x + 13 = 0$. **25.** $(2, 3)$; $(\frac{3}{2}, 4)$.

27. $(1\frac{3}{4}, -2\frac{1}{3})$; $(-1\frac{3}{4}, 2\frac{1}{3})$. **29.** $(-1\frac{3}{4}, -2.1)$; $(2\frac{1}{2}; -5\frac{1}{4})$.

31. $(\frac{2}{5}\sqrt{65}, \pm \frac{1}{5}\sqrt{35})$; $(-\frac{2}{5}\sqrt{65}, \pm \frac{1}{5}\sqrt{35})$.

33. $(\frac{2}{3}, -2)$; $(4, -7)$. **35.** 1. **37.** ± 2; $\pm \frac{1}{2}$. **39.** $x = 1$.

Exercise 38. Page 94

1. $\frac{6}{5}$. 3. $\frac{3}{4}$. 5. $\frac{z}{a^3}$. 7. $\frac{54}{13}$. 9. $y = -\frac{8}{21}$.
11. 72 and 54. 13. 5500 sq. in. 15. 28.7′.
17. ± 1. 19. $\pm 4i$. 21. $\pm \frac{y}{x^2}$.

Exercise 39. Page 96

1. $H = \frac{kx}{w^3}$. 3. $Z = \frac{k\sqrt{x}}{y^2}$. 5. $x + 2 = \frac{k}{y+3}$. 7. $w = \frac{k}{d^2}$.
9. y is proportional to w.
11. w is directly proportional to x and y^2 and inversely proportional to z.
13. $P = \frac{3}{8}x^2$. 15. $U = \frac{27xy}{2z^2}$. 17. $y = \frac{20}{3}$.
19. 784 ft. 21. 2700 lb. 23. 270 lb. 25. 8.82″.
29. A stone $5\frac{1}{3}$ ft. in diameter. 31. 47.43′.
33. $(32, -16, 40)$. 35. $(6, -2, 4)$ or $(-6, 2, -4)$.

Exercise 40. Page 100

1. $x^6 + 6x^5u + 15x^4u^2 + 20x^3u^3 + 15x^2u^4 + 6xu^5 + u^6$.
3. $c^4 - 4c^3d + 6c^2d^2 - 4cd^3 + d^4$.
5. $x^4 - 8x^3a + 24x^2a^2 - 32xa^3 + 16a^4$.
7. $64x^6 - 192x^5b + 240x^4b^2 - 160x^3b^3 + 60x^2b^4 - 12xb^5 + b^6$.
9. $x^4 + 2x^3b^2 + \frac{3}{2}x^2b^4 + \frac{1}{2}xb^6 + \frac{1}{16}b^8$.
11. $x^{12} - 6x^{10}y^3 + 15x^8y^6 - 20x^6y^9 + 15x^4y^{12} - 6x^2y^{15} + y^{18}$.
13. $x^{\frac{5}{2}} - 10x^2y^{\frac{1}{2}} + 40x^{\frac{3}{2}}y - 80xy^{\frac{3}{2}} + 80x^{\frac{1}{2}}y^2 - 32y^{\frac{5}{2}}$.
15. $r^{\frac{4}{3}} + 8r + 24r^{\frac{2}{3}} + 32r^{\frac{1}{3}} + 16$.
17. $x^4 - 4x^3y^{-1} + 6x^2y^{-2} - 4xy^{-3} + y^{-4}$.
19. $a^{\frac{5}{3}} - 10a^{\frac{4}{3}}x + 40ax^2 - 80a^{\frac{2}{3}}x^3 + 80a^{\frac{1}{3}}x^4 - 32x^5$.
21. $\frac{8}{x^3} - \frac{12}{bx^2} + \frac{6}{b^2x} - \frac{1}{b^3}$.
23. $\frac{a}{z^5} - \frac{5a^4}{yz^{\frac{7}{2}}} + \frac{10a^3}{y^2z^2} - \frac{10a^2}{y^3z^{\frac{1}{2}}} + \frac{5az}{y^4} - \frac{z^{\frac{5}{2}}}{y^5}$.
25. $a^{15} - 45a^{14} + 945a^{13}$. 27. $m^{40} + 60m^{38}n + 1710m^{36}n^2$.
29. $3^6 - 2916\sqrt{3} + 16{,}038$. 31. $a^{-8} + 8a^{-7}x^{-2} + 28a^{-6}x^{-4}$.
33. $x^n + nx^{n-1}y + \frac{n(n-1)}{2}x^{n-2}y^2$. 35. $z^h - hz^{h-1}w^2 + \frac{h(h-1)}{2}z^{h-2}w^4$.

Exercise 41. Page 102

1. $126x^5y^4$. 3. $70x^4z^{12}$. 5. $\frac{n(n-1)(n-2)(n-3)(n-4)}{5!}x^{n-5}y^5$.
7. $36w^7z^2$. 9. $56x^6y^5$. 11. $\frac{21}{16}x^3$. 13. .000028.

15. $-20x^3y^6$. **17.** $\frac{5}{2}x^{\frac{3}{2}}y^2$; $-\frac{5}{4}xy^3$. **19.** $35x^3y^{\frac{4}{3}}$. **21.** $20x^3y^3$.

23. $-35a^8b^3$; $35a^6b^4$. **25.** 96,059,601. **27.** 226,981. **29.** 1.219. **31.** .961.

Exercise 42. Page 105

1. 15; 18; 21; 24.
3. 17; 14; 11; 8.
5. $S = 645$; $l = 78$.
7. $S = -882$; $l = -72$.
9. $S = 46.86$; $l = .78$.
11. $d = 16$; $S = 5460$.
13. $S = 323$; $a = -19$.
15. $l = -99$; $d = -2$.
17. -1. **19.** (5, 8, 11, 14). **21.** (10.5, 6, 1.5, -3, -7.5, -12)
23. 22. **25.** 7. **27.** 21. **29.** 136th term
31. $\frac{45}{31}$. **33.** 36,270. **35.** 25,250. **37.** 425.
39. \$23,200. **41.** 3780. **43.** \$53,250.

Exercise 43. Page 108

1. (5, 15, 45, 135).
3. (16, 8, 4, 2).
7. (i) the result is a G.P. with $r = 3$.
9. 26,244.
11. $l = \frac{5}{6}$; $S = \frac{315}{16}$.
13. $l = -\frac{1}{2}$; $S = -\frac{11}{2}$.
15. $S = \dfrac{a - ax^{45}}{1 - x^3}$; $l = ax^{42}$.
17. 6558.
19. $\dfrac{(1.06)^{30} - (1.06)^4}{.06}$.
21. $\dfrac{1 - (1.02)^{-15}}{.02}$.
23. $\dfrac{(1.02)^{10} - 1}{(1.02)^{\frac{1}{2}} - 1}$.
25. $S = 242$; $n = 5$.
27. $a = 25$; $n = 5$.
29. $S = \frac{2047}{4}$; $n = 11$.
31. 1280.
33. (12; 36; 108).
35. (1; 10; 100; 1000; 10,000; 100,000).
37. $\sqrt{xy}$ or $-\sqrt{xy}$ according as x and y are positive or negative.
39. 10. **41.** $(\frac{5}{2}, \frac{5}{4}, \frac{5}{8})$. **43.** $\frac{635}{64}$. **45.** \$450; \$3600.

Exercise 44. Page 110

1. \$10,000. **3.** 2900 ft. **5.** \$80; \$200; \$500; \$1250; \$3125.
7. $\dfrac{x - x^{41}}{1 - x}$. **9.** \$11,360. **11.** 55.34 in. **17.** 852.
21. 135,760.57. **23.** 15.5%. **25.** 20%.

Exercise 45. Page 112

1. $\frac{1}{4}$; $\frac{1}{6}$; $\frac{1}{8}$; $\frac{1}{10}$.
3. $\frac{5}{12}$; $\frac{1}{2}$; $\frac{5}{8}$; $\frac{5}{6}$.
5. 1; $\frac{5}{7}$; $\frac{5}{9}$; $\frac{5}{11}$; $\frac{5}{13}$
7. 2.
9. 16.
11. 18.

Exercise 46. Page 115

1. 9. 3. $\frac{50}{9}$. 5. $\frac{10}{11}$. 7. $\frac{2}{3}$. 9. $\frac{13}{99}$.
11. $\frac{1}{99}$. 13. $\frac{47}{9}$. 15. $\frac{38}{11}$. 17. $\frac{2651}{4500}$. 19. $\frac{4000}{11}$.
21. $\frac{2410}{33}$. 23. $\frac{1}{13}$. 25. 280′. 27. $5\frac{1021}{1024}$; 6.

Exercise 47. Page 119

1. $6 < x$. 3. $12 > x$. 5. $-3 < x$. 7. $-2 < x$.
9. $\frac{5}{3} < x$. 11. $x < \frac{13}{45}$. 13. $5 < x$. 15. $x < 3$.
17. $2 < x < 9$. 19. $-5 < x < 5$. 21. $x < -3$ or $3 < x$.
23. $|x| < 6$; between -6 and 6. 25. $|x| > 7$. 27. $x < \frac{2}{5}$.
29. $x < -5$ and $x > 5$. 31. $-2 < x < 4$.
33. $x < -1$ and $x > 4$. 35. $x < -\frac{1}{4}$ and $x > 0$.
37. No solution. 41. $|x| > 10$. 43. $|x| \geqq 4$.
45. $x \leqq 2$ and $x \geqq 3$. 47. Real if $|k| \geqq 2$; imag. if $|k| < 2$.
49. (a) $k = \pm\frac{3}{4}$; (b) $|k| < \frac{3}{4}$; (c) $|k| > \frac{3}{4}$.
51. (a) $k = \pm\sqrt{10}$; (b) $|k| < \sqrt{10}$; (c) $|k| > \sqrt{10}$.

Exercise 49. Page 122

1. $5i$. 3. $8ix$. 5. $\pm 9i$. 7. $\pm 5i\sqrt{3}$. 9. $\pm\frac{3}{4}ix$.
11. -1. 13. $-i$. 15. $-15i$. 17. $243i$. 19. $26 + 7i$.
21. -9. 23. $-4 + 2i$. 25. 6. 27. $x = 2$; $y = -3$.
29. $x = 4$; $y = 0$. 31. $x = 2$; $y = \frac{5}{2}$. 33. $-\frac{3}{25} - \frac{22}{25}i$.
35. $\frac{5}{2} - \frac{7}{2}i$. 37. $-\frac{8}{13} + \frac{12}{13}i$. 39. $-\frac{2}{13} - \frac{23}{13}i$.
41. $-\frac{5}{2}i$. 43. $-i$. 45. 729. 47. $-11 - 2i$.
49. $\frac{3}{34} + \frac{5}{34}i$. 51. $\frac{1}{2}i$. 53. $a - bi$.

Exercise 50. Page 125

17. $5 + 5i$. 19. $5 + 2i$. 21. $7 + 3i$. 23. $12i - 3$. 25. $1 + 2i$.

Exercise 51. Page 127

1. $\frac{5}{2} + \frac{5}{2}i\sqrt{3}$. 3. $4 + 0i$. 5. $6 + 0i$. 7. $\sqrt{2} - i\sqrt{2}$.
9. $-.839 + .545i$. 11. $5\sqrt{2}(\cos 45° + i \sin 45°)$.
13. $6(\cos 0° + i \sin 0°)$. 15. $3(\cos 270° + i \sin 270°)$.
17. $2(\cos 30° + i \sin 30°)$. 19. $2(\cos 240° + i \sin 240°)$.
21. $5(\cos 323.1° + i \sin 323.1°)$. 23. $\cos 330° + i \sin 330°$.
25. $\sqrt{2}(\cos 45° + i \sin 45°)$; $\sqrt{2}(\cos 315° + i \sin 315°)$.
27. $2(\cos 120° + i \sin 120°)$; $2(\cos 240° + i \sin 240°)$.
29. (1) 180°; (2) 270°.

Exercise 52. Page 129

1. $3 + 3i\sqrt{3}$. 3. $6(\cos 20° + i \sin 20°)$. 5. $4 + 4i\sqrt{3}$.
7. $-8 + 8i\sqrt{3}$. 9. $324(\cos 180° + i \sin 180°)$, or -324.

11. $16 + 16i\sqrt{3}$. 13. $-1 + i\sqrt{3}$. 15. $12(\cos 130° + i \sin 130°)$.

17. $\frac{3}{3}\sqrt{3} - \frac{3}{3}i$.

Exercise 53. Page 132

1. $2(\cos 25° + i \sin 25°)$; $2(\cos 115° + i \sin 115°)$; etc.
3. $(\frac{3}{2}\sqrt{2} + \frac{3}{2}i\sqrt{2})$; $3(\cos 165° + i \sin 165°)$; $3(\cos 285° + i \sin 285°)$.
5. 2; $(-1 + i\sqrt{3})$; $(-1 - i\sqrt{3})$. 7. $(2\sqrt{2} + 2i\sqrt{2})$; $(-2\sqrt{2} - 2i\sqrt{2})$.
9. 2; $2(\cos 72° + i \sin 72°)$; $2(\cos 144° + i \sin 144°)$; etc.
11. $(\sqrt{2} \pm i\sqrt{2})$; $(-\sqrt{2} \pm i\sqrt{2})$.
13. $2(\cos 105° + i \sin 105°)$; $(-\sqrt{2} - i\sqrt{2})$; $2(\cos 345° + i \sin 345°)$.
15. $\sqrt{5}(\cos 76.7° + i \sin 76.7°)$; $\sqrt{5}(\cos 166.7° + i \sin 166.7°)$; etc.
17. 3; $3(\cos 72° + i \sin 72°)$; $3(\cos 144° + i \sin 144°)$; etc.
19. 1; $(\cos 40° + i \sin 40°)$; $(\cos 80° + i \sin 80°)$; $\cdots$; $(\cos 320° + i \sin 320°)$.

Exercise 54. Page 134

1. $R = 29$. 3. $R = 48$. 5. Yes; $x^2 - 3x + 1$.

7. Yes; $x^2 + 2x + 4$. No. 9. $-\frac{4}{3}$; $\frac{1}{3}$.

Exercise 55. Page 136

1. $2x^2 + 6x + 20 + \dfrac{65}{x-3}$. 3. $3x + 8 + \dfrac{29}{x-3}$.

5. $2x^2 + 7x + 12 + \dfrac{19}{x-2}$. 7. $3x^2 - 2x + 4$. 9. $2x^2 + 2x - 3 + \dfrac{9}{2x+1}$.

11. 68; -292. 13. 7; -84. 15. $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$.

Exercise 56. Page 138

11. 4.2; -1.2. 13. 4; 1.4; -1.5. 15. $.17$.

Exercise 57. Page 141

1. 2; -5; -7. 3. 0; -4; $\frac{1}{6}(-7 \pm i\sqrt{59})$.

5. $x^4 + 3x^3 - 25x^2 - 39x + 180 = 0$.

7. $12x^4 + 7x^3 - 28x^2 + 12x = 0$. 9. $x^4 + x^2 - 12 = 0$.

11. $9x^4 - 12x^3 + 148x^2 - 192x + 64 = 0$.

13. $x^4 - 4x^3 + 2x^2 - 4x + 1 = 0$. 15. $x^3 - 6x^2 + 15x - 14 = 0$.

17. $x^3 + 6x^2 + 12x + 8 = 0$. 19. $9x^4 - 54x^3 + 80x^2 + 6x - 9 = 0$.

21. $-2 - 4i$. 23. $-3 + 4i$. 25. $x^4 - 8x^3 + 27x^2 - 38x + 26 = 0$.

Exercise 59. Page 144

1. $2x^3 + 5x^2 + 3x + 5 = 0$. 3. $4x^4 - 3x^3 - 2x - 4 = 0$.

5. $x^3 - 3x - 5 = 0$. 7. $3x^5 + 2x^4 + 5x = 7$.

Exercise 60. Page 145

1. One pos. and two imag.
3. (*a*) Four imag. or (*b*) two imag. and two pos.
5. (*a*) One pos., one neg., and two imag. or (*b*) four imag. From a very rough graphical consideration, (*b*) is ruled out and hence (*a*) is true.
7. (*a*) One pos., one neg., and two imag. or (*b*) four imag. A rough graph rules out (*b*) and hence (*a*) is true.
9. (*a*) One pos., one neg., and four imag. or (*b*) six imag. A rough graph rules out (*b*) and hence (*a*) is true.
11. (*a*) One pos., one neg., and two imag. or (*b*) four imag. A rough graph rules out (*b*).
13. Six imag.
15. (*a*) One pos. and six imag., or (*b*) one neg. and six imag., or (*c*) two pos., one neg., and four imag. A rough graph shows one neg. root. Hence, (*b*) and (*c*) are the only possibilities.
17. Three pos.
19. Two pos. and three neg.
21. Two imag.
23. Four imag.

Exercise 61. Page 147

1. Upper = 3; lower = − 5.
3. Upper = 6; lower = 0.
5. Upper = 4; lower = − 1.

Exercise 62. Page 149

1. 1; − 2; − 3.
3. − 2; ± 3.
5. − 2; $\frac{1}{2}(3 \pm i\sqrt{3})$.
7. 7; 1; − 2; − 6.
9. None.
11. $\frac{5}{2}$; $(-1 \pm i)$.
13. $\frac{1}{4}$; $(1 \pm i\sqrt{2})$.
15. $\pm \frac{2}{3}$; $\frac{1}{2}(1 \pm i\sqrt{3})$.
17. 1; $\frac{1}{2}$; $-\frac{1}{8}$.
19. − 7.
21. None.
23. 1.

Exercise 63. Page 150

1. $3x^3 + 4x^2 - 20x + 24 = 0$.
3. $x^4 - x^3 + 6x^2 - 40 = 0$.
13. $\frac{3}{4}$; $-\frac{1}{2}$; $\frac{3}{2}$.
15. None.

Exercise 64. Page 153

Note. For the convenience of the student, results are given more accurately than requested in the text.

1. 1.213.
3. .802.
5. 2.357; 2.692.
7. 2.154.
9. − 2; − .150; − 1.724; 3.874.
11. ± 1.720.
13. 1.427; 1.757; − 5.184.
15. 6.350.
17. − 2.770.
19. − .50; .63; 2.34.
21. − 8.22; 3.43.
23. (1.82, .66); (2.63, 2.46).
25. 1.37″ or 2.61″.
27. 1.77″.
29. 637°.

Exercise 65. Page 155

1. $2x^3 + 8x^2 + 5x + 42 = 0$.
3. $x^4 + 1.2x^3 - 2.46x^2 + 3.308x = 5.7619$.
5. $3x^3 + 1.73x^2 - 2.1119x + 5.061719 = 0$.

Exercise 66. Page 159

3. $x^4 + 2x^3 - 5x^2 - 4x + 6 = 0$.
5. $-\frac{10}{3}$. **7.** 3; -3; -4. **9.** 1; 2; 4.

Exercise 67. Page 162

1. -2; $(1 \pm 2i\sqrt{3})$. **3.** 4.13; -9.52; $-.61$.
5. $\frac{1}{2}(-1 \pm i\sqrt{3})$; $\frac{1}{2}(3 \pm i\sqrt{3})$. **7.** -3; -3; -4; 1.

Exercise 68. Page 165

1. 10^6. **3.** 10^{-3}. **5.** 10^0. **7.** 15.326; 15.3.
9. .31427; .314. **11.** 195.64; 196. **13.** .034564; .0346.
15. 38,500. **17.** .0001935. **19.** $3.807(10^6)$.
21. $3.57(10^{-4})$. **23.** 566.5 and 567.5. **25.** 566.95 and 567.05.
27. 8.1385 and 8.1395. **29.** 31.54; .586. **31.** 3.8; .78.
33. $9.3250(10^6)$; $9.32(10^6)$. **35.** $2.3500(10^7)$; $2.35(10^7)$.

Exercise 69. Page 167

1. $\frac{1}{a^5}$. **3.** $\frac{1}{16}$. **5.** $\frac{1}{100}$. **7.** $a^{\frac{1}{2}}$. **9.** $5^{\frac{1}{3}}$. **11.** $10^{\frac{1}{4}}$.
13. $5 = \log_3 N$. **15.** $\log_{10} N = -3$. **17.** $\log_{10} N = -.4$.
19. $\log_4 64 = 3$. **21.** $\log_3 27 = 3$. **23.** $\log_2 \frac{1}{32} = -5$.
25. 25. **27.** 216. **29.** 1. **31.** $\frac{1}{16}$. **33.** 2. **35.** 5.
37. 2. **39.** $\frac{1}{2}$. **41.** 3. **43.** 2. **45.** $\frac{1}{2}$. **47.** -4.
49. $a = 4$. **51.** $a = 10$. **53.** $a = 27$. **55.** $a = 25$.
57. $N = 729$. **59.** $N = 10$. **61.** $a = 100$. **63.** $a = \frac{1}{4}$.

Exercise 70. Page 170

1. .7781. **3.** 1.4771. **5.** .3680. **7.** .5441.
9. $-.1549$. **11.** 1.4065. **13.** .9542. **15.** 1.4313.
17. .2817. **19.** $-.4771$. **21.** .3820. **23.** $-.2720$.

Exercise 71. Page 174

1. Ch. = 2; man. = .9356. **3.** Ch. = 15; man. = .2162.
5. Ch. = -2; man. = .700. **7.** Ch. = -3; man. = .2356.
9. 9.2562 − 10. **11.** 4.4932 − 10. **13.** 5. **15.** -4.
17. -6. **19.** 1.6355. **21.** 7.8949 − 10. **23.** 0.9759.

25. 4.2504. 27. 8.9345 − 10. 29. 5.0043. 31. 5.1959.
33. 243. 35. 4660. 37. 1.43. 39. 74.0.
41. .302. 43. .00589. 45. .0960. 47. .000900.
49. .00500; .0200. 51. 2.29820. 53. 0.14270. 55. 8.04844 − 10.
57. 9.40875 − 10. 59. 4.77525. 61. 6.25527 − 10. 63. 9.90363 − 10.
65. 3.61993. 67. 6.00000. 69. 0.69897. 71. 17.83.
73. 48,460. 75. .2674. 77. .003097. 79. 4.577.
81. 4159. 83. .0003600. 85. 1.299.

Exercise 72. Page 177

1. 3.2615. 3. 2.7261. 5. 1.5556. 7. 9.4790 − 10.
9. 9.7503 − 10. 11. 8.1939 − 10. 13. 4.9546. 15. 7.1581 − 10.
17. 6.0910 − 10. 19. 6.4950. 21. 1725. 23. $1.459(10^6)$.
25. 1379. 27. 39.95. 29. .0002162. 31. .4693.
33. 7695. 35. 1.030. 37. .00009738. 39. .4236.
41. 4.26865. 43. 0.72605. 45. 9.47898 − 10. 47. 0.67374.
49. 8.86666 − 10. 51. 9.87715 − 10. 53. 7.60274 − 10. 55. 6.89570 − 10.
57. 5.98538. 59. 9.78533 − 20. 61. 163.64. 63. 21.747.
65. .45007. 67. .0087124. 69. $(1.0782)10^6$. 71. 9.3402.
73. .10200. 75. $1.7398(10^{-14})$.

Exercise 73. Page 179

Note. Results obtained by use of 5-place logarithms are given in blackface type in the remainder of this chapter.

1. 24.91; **24.909.** 3. .07942; **.079410.** 5. .2009; **.20086.**
7. − .007667; **− .0076660.** 9. 51.10; **51.098.** 11. .8142; **.81422.**
13. .1406; **.14061.** 15. .003069; **.0030681.** 17. 5542; **5544.4.**
19. $1.047(10^4)$; **10.464.** 21. 27.61; **27.609.**
23. − $2.627(10^{-8})$; **− $2.6266(10^{-8})$.** 25. $1.580(10^{-5})$; **$1.5802(10^{-5})$.**
27. 38.96; **38.955.** 29. (*a*) $4.792(10^5)$; **$4.7922(10^5)$**: (*b*) 8.065; **8.0662.**
31. (*a*) $5.616(10^{-4})$; **$5.6160(10^{-4})$**: (*b*) − .1626; **− .16263.**

Exercise 74. Page 181

1. 5358; **5359.5.** 3. .4107; **.41082.** 5. 1.044; **1.0440.**
7. .9500; **.94986.** 9. 1.315; **1.3158.** 11. .6030; **.60296.**
13. 28.93; **28.935.** 15. .1585; **.15849.** 17. − 1.010; **− 1.0099.**
19. 50.32; **50.324.** 21. 41.47; **41.470.** 23. .1266; **.12658.**
25. 2.111; **2.1111.** 27. 1.041; **1.0412.** 29. .8630; **.86258.**
31. 50.12; **50.466.** By preliminary use of 7-place table, the results are .5050; **.50504.**
33. 141.9; **141.82.** 35. 215.1; **215.08.** 37. .4971; **.49714.**
39. .001352; **.0013525.** 41. .9388; **.93896.** 43. .3986; **.39882.**

45. -1.916; $\mathbf{-1.9156}$. **47.** 21.76; **21.758.** **49.** 134.9; **134.84.**
51. -136;* $\mathbf{-.1366}$.* **53.** 1.118; **1.1177.** **55.** 4.908; **4.9086.**
57. .1730; **.17294.**
59. By 4-place table; (*a*) $2.219(10^4)$; (*b*) $\mathbf{3.222(10^{-5})}$.
61. .02323; **.023229.** **63.** .0007867; **.0007869.***
65. (*a*) .09960; **.099598** (sec.): (*b*) 1396; **1396.3** (cm.).
67. 145.6; **145.56** (lb. per sec.).

Exercise 75. Page 184

1. 1.341; **1.3410.** **3.** 1.319; **1.3194.** **5.** -5.195; $\mathbf{-5.1923}$.
7. ± 1.100; $\mathbf{\pm 1.1001}$. **9.** 11.3; **11.30.**
11. 2.617; $-.617$;* **2.6165**; $\mathbf{-.6165}$.*
13. $1.153(10^8)$; $\mathbf{1.1535(10^8)}$. **15.** 5.63;* **5.634.***
17. (*a*) 11.72; **11.722** (mm.): (*b*) 613.9°; **613.91°.**
19. (*a*) 11.06; **11.067** (cm.): (*b*) .842; **.8412** (cm.).

Exercise 76. Page 185

1. 4.317; **4.3176.** **3.** 1.291; **1.2911.** **5.** 2.303; **2.3026.**
7. -1.449; $\mathbf{-1.4496}$. **9.** -14.2;* $\mathbf{-14.20}$.*
11. (*a*) 8.382; **8.3822**: (*b*) 1.474; **1.4743.**

Exercise 80. Page 193

1. 360. **3.** 32. **5.** 24. **7.** 24. **9.** 120; 60. **11.** 20,160.
13. 216. **15.** (1) 2520; (2) 360; (3) 720; (4) 1440. **17.** 600.
19. 2400. **21.** 144. **23.** 144. **25.** 60. **27.** 7200. **29.** 72.

Exercise 81. Page 197

1. 20. **3.** 6. **5.** 1320. **7.** 4200. **9.** 30,240. **11.** 60.
13. 420. **15.** (*a*) 5040; (*b*) 40,320.
17. (*a*) 360; (*b*) 60; (*c*) 120; (*d*) 120. **19.** 48. **21.** 1440.
23. 2880. **25.** (*a*) 3600; (*b*) 1440. **27.** (*a*) 14,400; (*b*) 2880.
29. 336. **31.** 483,840.

Exercise 82. Page 201

1. (*a*) 4; (*c*) 6 **3.** 21. **5.** 20. **7.** 4. **9.** 20.
11. (*a*) 105; (*b*) 2450; (*c*) 110. **13.** 1,127,251. **15.** 5040.

Exercise 83. Page 203

1. (*a*) 20; (*b*) 42. **3.** 81. **5.** 1440. **7.** 60,480.
9. (*a*) 840; (*b*) 1316. **11.** 10,080. **13.** 27,720. **15.** 1880.
17. 1152. **19.** 91. **21.** 151,200. **23.** (*a*) 21; (*b*) 64.

* The computation does not yield reliable results beyond the last digit given in the answer.

25. 480. **27.** 2520. **29.** (*a*) 35; (*b*) 64. **31.** 132.
33. 252,000. **35.** 21,000. **37.** 1728. **39.** 240.
41. Check the answer. **43.** Check the answer.

Exercise 84. Page 206

1. $x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7.$
3. $x^{10} - 10x^8y^3 + 40x^6y^6 - 80x^4y^9 + 80x^2y^{12} - 32y^{15}.$
5. $-126x^4y^5.$ **7.** $165w^3z^4.$
9. $x^{14} + {}_{14}C_1x^{13}y + {}_{14}C_2x^{12}y^2 + \cdots + {}_{14}C_{13}xy^{13} + y^{14}.$ **11.** 255.

Exercise 85. Page 210

1. (*a*) $\frac{2}{3}$; (*b*) $\frac{1}{3}$. **5.** (*a*) $\frac{1}{13}$; (*b*) $\frac{2}{13}$.
7. (*a*) $\frac{2}{5}$; (*b*) $\frac{3}{5}$; (*c*) $40. **9.** (*a*) $\frac{2}{21}$; (*b*) $\frac{3}{7}$; (*c*) $\frac{10}{21}$.
11. (*a*) $\frac{10}{143}$; (*b*) $\frac{60}{143}$. **13.** $\frac{91{,}192}{98{,}505}$. **15.** $\frac{86{,}878}{94{,}089}$.
17. (*a*) $\frac{721}{91{,}192}$; (*b*) $\frac{729}{91{,}192}$. **19.** (*a*) $\frac{2}{13}$; (*b*) $\frac{30}{91}$; (*c*) $\frac{5}{13}$; (*d*) $\frac{19}{91}$.
21. $\frac{1}{5}$. **23.** $\frac{44}{4165}$. **25.** $\frac{24}{49}$. **27.** $\frac{10}{63}$.
29. (*a*) $\frac{1}{36}$; (*b*) $\frac{5}{108}$; (*c*) $\frac{5}{54}$. **31.** (*a*) $\frac{11}{24}$; (*b*) $\frac{7}{30}$; (*c*) $\frac{7}{15}$.
33. $\frac{253}{715}$; $\frac{156}{715}$. **35.** (*a*) $\frac{1}{64}$; (*b*) $\frac{5}{16}$. **37.** $\frac{1}{10}$.
39. In his 74th year; $\frac{2505}{92{,}367}$. **41.** $\frac{2}{5}$. **43.** $\frac{11}{4165}$.

Exercise 86. Page 216

1. (*a*) $\frac{1}{15}$; (*b*) $\frac{4}{15}$. **3.** $\frac{1}{12}$. **5.** (*a*) $\frac{1}{12}$; (*b*) $\frac{1}{3}$. **7.** $\frac{1}{216}$.
9. (*a*) $\frac{2}{9}$; (*b*) $\frac{4}{9}$. **11.** $\frac{4}{25}$. **13.** $\frac{31}{80}$. **15.** (*a*) $\frac{3}{14}$; (*b*) $\frac{11}{28}$
17. (*a*) $\frac{1}{64}$; (*b*) $\frac{1}{16}$; (*c*) $\frac{8}{2197}$. **19.** $\frac{5}{9}$.
21. A, $\frac{1}{8}$; B, $\frac{1}{8}$; C, $\frac{1}{4}$; D, $\frac{1}{2}$. **23.** $\frac{2}{5}$.

Exercise 87. Page 219

1. (*a*) $\frac{128}{625}$; (*b*) $\frac{821}{3125}$; (*c*) $\frac{2944}{3125}$. **3.** (*a*) $\frac{15}{64}$; (*b*) $\frac{57}{64}$; (*c*) $\frac{11}{32}$.
5. (*a*) $\frac{80}{243}$; (*b*) $\frac{64}{81}$. **7.** $\frac{81}{10{,}000}$. **9.** $\frac{2}{225}$.
11. $\dfrac{(65706)^2(137841)}{(89751)^3}$ or .823. **13.** $\frac{3125}{7776}$.

Exercise 89. Page 228

1. 5. **3.** 3.

Exercise 90. Page 234

1. For 2d column:

$$-m_1\begin{vmatrix} c_2 & v_2 \\ c_3 & v_3 \end{vmatrix} + m_2\begin{vmatrix} c_1 & v_1 \\ c_3 & v_3 \end{vmatrix} - m_3\begin{vmatrix} c_1 & v_1 \\ c_2 & v_2 \end{vmatrix}.$$

3. 42. **5.** -63. **7.** $xy(x-1)(y-1)(y-x)$. **9.** 60.
11. 146. **13.** -35. **15.** $(x-y)(y-w)(w-x)$.

Exercise 91. Page 238

1. $(1, 2, -2)$. **3.** $(2, -4, \frac{1}{2})$. **5.** $(2, -\frac{1}{2}, 1, 2)$.
7. $(2, -1, -1, 2)$. **9.** $(\frac{1}{2}, 1, -1, 2, 2)$.

Exercise 92. Page 241

1. Nontrivial solutions exist. **3.** No nontrivial solutions.
11. $1;\ -\frac{14}{9}$. **13.** ± 2.

Exercise 93. Page 246

1. $\dfrac{4}{x-4} - \dfrac{3}{2x+1}$. **3.** $\dfrac{1}{x-7} + \dfrac{1}{x+2}$.

5. $\dfrac{2}{y} + \dfrac{2}{y+3} - \dfrac{4}{(y+3)^2}$. **7.** $\dfrac{2}{(x-1)^3} + \dfrac{2}{(x-1)^2} + \dfrac{1}{x-1}$.

9. $\dfrac{1}{(x-1)^2} + \dfrac{2}{x-1} + \dfrac{3}{x-2}$. **11.** $1 - \dfrac{4}{2-x} + \dfrac{1}{(2-x)^2} + \dfrac{1}{x-1}$.

Exercise 94. Page 247

1. $\dfrac{4}{2x-5} + \dfrac{3-x}{x^2+2}$. **3.** $\dfrac{3}{x-2} + \dfrac{x-1}{x^2+2x+4}$.

5. $\dfrac{3x+1}{x^2+3} - \dfrac{1}{x+1} + \dfrac{2}{(x+1)^2}$. **7.** $\dfrac{3x}{x^2+5} + \dfrac{2x+1}{2x^2+3}$.

9. $\dfrac{2}{1-5x} + \dfrac{1}{1+3x} + \dfrac{3}{(1+3x)^2}$. **11.** $-\dfrac{1}{3(2x+1)} + \dfrac{2x+1}{3(4x^2-2x+1)}$.

13. $\dfrac{1}{x^3} - \dfrac{2}{x^2} + \dfrac{1}{x} + \dfrac{x+5}{x^2+1}$.

15. $\dfrac{9x-8}{3(x^2+x+4)} - \dfrac{6x+5}{3(x^2+2x+5)}$. **17.** $\dfrac{x+1}{x^2+2} - \dfrac{2x}{(x^2+2)^2}$.

19. $\dfrac{5}{x^2+3x+1} + \dfrac{3x-7}{(x^2+3x+1)^2}$.

21. $\dfrac{1}{(x-1)^2} - \dfrac{2}{2x^2+x+1} - \dfrac{4x}{(2x^2+x+1)^2}$.

23. $\dfrac{1}{(3x+2)^2} - \dfrac{2}{3x+2} + \dfrac{3}{(x-1)^2} - \dfrac{2}{x-1}$.

25. $\dfrac{5}{(2x-1)^2} - \dfrac{2}{2x-1} + \dfrac{3}{(x-3)^2} - \dfrac{.5}{x-3}$.

Exercise 95. Page 254

1. 1.037. **3.** .947. **5.** 1.649. **7.** .479. **9.** .100.

Note 1. In Problems 11–19, the answers are given correct to three decimal places for use in case the instructor desires to request greater accuracy than specified on page 255.

11. 1.049. **13.** .971. **15.** 1.013. **17.** 10.488. **19.** 2.289.

Exercise 97. Page 259

Note 1. Converges and diverges are abbreviated by C and D, respectively.

21. C. **23.** D. **25.** D. **27.** C. **31.** C.

Exercise 98. Page 263

1. 3. **3.** $\frac{1}{2}$. **5.** C. **7.** D, by comparison test.
9. C. **11.** C. **13.** D. **15.** D. **17.** C.
19. C. **21.** D. **23.** C. **25.** C.

Exercise 99. Page 269

1. C. **3.** C. **5.** C. **7.** D. **9.** D.
11. $-1 \leqq x < 1$. **13.** $-4 < x < 4$. **15.** $-\sqrt{2} < x < \sqrt{2}$.
17. $-\infty < x < \infty$; i.e., converges for all values of x. **19.** $-2 < x < 2$.
21. $-3 \leqq x \leqq 3$. **23.** $-1 < x < 1$. **25.** $-1 < x < 1$.
27. $-3 \leqq x - 2 \leqq 3$, or $-1 \leqq x \leqq 5$.

Exercise 100. Page 275

1. -8. **3.** $-\frac{23}{4}$. **5.** -1.8.
7. $1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 225$. **9.** $x_1 + x_2 + x_3 + x_4$.
11. $b_1c_1 + b_2c_2 + \cdots + b_nc_n$.
13. $\frac{1}{2}(1^2) + \frac{1}{2}(2^2) + \frac{1}{2}(3^2) + \cdots + \frac{1}{2}(7^2) = 70$.
15. $5 + 8 + 11 + 14 = 38$.
17. $\Sigma xy = x_1y_1 + x_2y_2 + \cdots + x_ny_n$;
$\Sigma x^2y^2 = x_1^2y_1^2 + x_2^2y_2^2 + \cdots + x_n^2y_n^2$;
$\Sigma(x - y) = (x_1 - y_1) + (x_2 - y_2) + \cdots + (x_n - y_n)$.
19. $\Sigma_{i=1}^{6} v_i^3$. **21.** $\Sigma_{i=1}^{n} 3x_i^2$. **23.** $A = 4$; $\sigma = 4$. **25.** $A = 3.8$; $\sigma = .36$.
27. $A = 1325\frac{1}{8}$; $\sigma = 4.045$. **29.** $A = 3.174$; $\sigma = .02166$.

Exercise 101. Page 278

1. $(1.90, -2.90)$. **3.** $(1.21, 1.39)$. **5.** $(.492, 2.60, .0634)$.

7. $a = \dfrac{5\Sigma xy - (\Sigma x)(\Sigma y)}{5\Sigma x^2 - (\Sigma x)^2}$; $b = \dfrac{(\Sigma y)(\Sigma x^2) - (\Sigma x)(\Sigma xy)}{5\Sigma x^2 - (\Sigma x)^2}$.

Exercise 102. Page 282

1. $y = 3x - 7$. **3.** $y = -3x^2 + x + 5$.
5. $y = -.4094x + 3.035$. **7.** $y = .5726x - .2308$. **9.** $y = .5397x$.
11. $y = 31.36t + 466.8$; in 1920, about $31.0(10^6)$.
13. $y = .2667x^2 - .0267x + .8667$. **15.** $y = -1.394t + 86.982$.
17. $y = 1.227x^2 + 1.682x - 6.091$; $y = x^3 - 3x - 5$.
19. $y = 2.3 \log x - .7$.

Exercise 103. Page 288

1. For y on x: $y = .367x + .633$.
   For x on y: $x = .786y + .214$. $r = .537$.

3. For y on x: $y = \frac{3}{4}x + 2$.
   For x on y: $x = .973y - 1.946$. $r = .854$.

5. .993.

7. .055.

Exercise 104. Page 292

1. $y = 3(2^x)$.

3. $y = 2.14(2.96^x)$.

5. $y = 664(.390^x)$.

7. $y = 3.70x^{2.50}$.

9. $y = .0303x^{2.001}$.

11. $y = 10.57(1.650^t)$ where $t = (\text{year} - 1900) \div 10$.

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